FLABloM: Functional linear algebra with block matrices

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Functional linear algebra with block matrices

- Inspired by work on parallel parsing by Bernardy & Jansson
- Matrices in Agda
- Reflexive, transitive closure of matrices

$$\begin{bmatrix} 1 & \begin{bmatrix} 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

Matrices

Desirable:

- Easy to program with
- Easy to write proofs with

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Possibilities:

- Vectors of vectors: $Vec\ (Vec\ a\ n)\ m$
- Functions from indices: $Fin\ m \rightarrow Fin\ n \rightarrow a$
- ...

Matrices: shapes

A type for shapes:

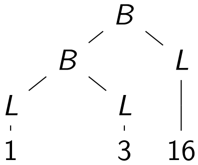
$$
\begin{aligned}
&\textbf{data}\ \mathit{Shape}\ :\ \mathit{Set}\ \textbf{where} \\
&\quad L\ :\ \mathit{Shape} \\
&\quad B\ :\ (s_1\ s_2\ :\ \mathit{Shape}) \rightarrow \mathit{Shape}
\end{aligned}
$$

Matrices: shapes

A type for shapes:

```
data Shape : Set where
  L : Shape
  B : (s₁ s₂ : Shape) → Shape
```

Shapes for one dimension: (a vector/row matrix)



Matrices: building blocks

Matrices are indexed by two shapes:

$$\textbf{data}\ M\ (a\ :\ Set)\ :\ (rows\ cols\ :\ Shape) \rightarrow Set$$

Matrices: building blocks

Matrices are indexed by two shapes:

$$\textbf{data}\ M\ (a\ :\ Set)\ :\ (rows\ cols\ :\ Shape) \to Set$$

$$\begin{array}{cccc}
[a]\,, & \big[\ [\cdots]\quad[\cdots]\ \big]\,, & \begin{bmatrix} \begin{bmatrix}\vdots\end{bmatrix} \\ \begin{bmatrix}\vdots\end{bmatrix} \end{bmatrix}, & \begin{bmatrix} \begin{bmatrix}\ddots\end{bmatrix} & \begin{bmatrix}\ddots\end{bmatrix} \\ \begin{bmatrix}\ddots\end{bmatrix} & \begin{bmatrix}\ddots\end{bmatrix} \end{bmatrix} \\
M\ a\ L\ L & M\ a\ L\ (B\ c_1\ c_2) & M\ a\ (B\ r_1\ r_2)\ L & M\ a\ (B\ r_1\ r_2)\ (B\ c_1\ c_2)
\end{array}$$

Matrices: a datatype

$$
\begin{array}{lll}
\multicolumn{3}{l}{\textbf{data}\ M\ (a : Set) : (rows\ cols : Shape) \to Set\ \textbf{where}} \\
\quad One & : & a \to M\ a\ L\ L \\
\quad Col & : & \{r_1\ r_2 : Shape\} \to \\
 & & M\ a\ r_1\ L \to M\ a\ r_2\ L \to M\ a\ (B\ r_1\ r_2)\ L \\
\quad Row & : & \{c_1\ c_2 : Shape\} \to \\
 & & M\ a\ L\ c_1 \to M\ a\ L\ c_2 \to M\ a\ L\ (B\ c_1\ c_2) \\
\quad Q & : & \{r_1\ r_2\ c_1\ c_2 : Shape\} \to \\
 & & M\ a\ r_1\ c_1 \to M\ a\ r_1\ c_2 \to \\
 & & M\ a\ r_2\ c_1 \to M\ a\ r_2\ c_2 \to \\
 & & M\ a\ (B\ r_1\ r_2)\ (B\ c_1\ c_2)
\end{array}
$$

Rings

A hierarchy of rings as Agda records:

- *SemiNearRing*
  $\simeq$, $+$, $\cdot$, *0* (+ is associative and commutes, 0 identity of $+$ and zero of $\cdot$, $\cdot$ distributes over $+$)
- *SemiRing*
  1 (1 identity of $\cdot$, $\cdot$ is associative)
- *ClosedSemiRing*
  an operation * with $w^* \simeq 1 + w \cdot w^*$.

Lifting matrices

We take a semi-(near)-ring and lift it to square matrices.
A lifting function *Square* for each *Shape* and ring structure.

$$
\begin{aligned}
\mathit{Square} &: \mathit{Shape} \to \mathit{SemiNearRing} &&\to \mathit{SemiNearRing} \\
\mathit{Square}' &: \mathit{Shape} \to \mathit{SemiRing} &&\to \mathit{SemiRing} \\
\mathit{Square}'' &: \mathit{Shape} \to \mathit{ClosedSemiRing} &&\to \mathit{ClosedSemiRing}
\end{aligned}
$$

Lifting matrices

(Parts of) lifted equivalence:

$$
\begin{array}{l}
\_\simeq_S\_ \ : \ \forall\ \{r\ c\} \to M\ s\ r\ c \to M\ s\ r\ c \to Set \\
(One\ x) \quad\quad \simeq_S \ (One\ x_1) \quad = \ x \ \simeq_s \ x_1 \\
(Row\ m\ m_1) \ \simeq_S \ (Row\ n\ n_1) \ = \ (m \ \simeq_S \ n) \times (m_1 \ \simeq_S \ n_1)
\end{array}
$$

(Parts of) lifted multiplication:

$$
\begin{array}{l}
\_\cdot_S\_ \ : \ \forall\ \{r\ m\ c\} \to M\ s\ r\ m \to M\ s\ m\ c \to M\ s\ r\ c \\
One\ x \quad\quad \cdot_S \ One\ y \quad = \ One\ (x \ \cdot_s \ y) \\
Row\ m_0\ m_1 \ \cdot_S \ Col\ n_0\ n_1 \ = \ m_0 \ \cdot_S \ n_0 \ +_S \ m_1 \ \cdot_S \ n_1
\end{array}
$$

Proofs: reflexivity

$$
\begin{array}{lcl}
\mathit{reflS} \;:\; \forall\, \{r\ c\} \rightarrow (X \;:\; M\ s\ r\ c) \rightarrow X \;\simeq_S\; X & & \\
\mathit{reflS}\ (\mathit{One}\ x) & = & \mathit{refl}_s\ \{x\} \\
\mathit{reflS}\ (\mathit{Row}\ X\ X_1) & = & \mathit{reflS}\ X\ ,\ \mathit{reflS}\ X_1 \\
\mathit{reflS}\ (\mathit{Col}\ X\ X_1) & = & \mathit{reflS}\ X\ ,\ \mathit{reflS}\ X_1 \\
\mathit{reflS}\ (Q\ X\ X_1\ X_2\ X_3) & = & \mathit{reflS}\ X\ ,\ \mathit{reflS}\ X_1\ , \\
 & & \mathit{reflS}\ X_2\ ,\ \mathit{reflS}\ X_3
\end{array}
$$

Closure for matrices

Computing the reflexive, transitive closure:

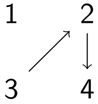
$$[a]^* = [a^*]$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^* = \begin{bmatrix} A_{11}^* + A_{11}^* \cdot A_{12} \cdot \Delta^* \cdot A_{21} \cdot A_{11}^* & A_{11}^* \cdot A_{12} \cdot \Delta^* \\ \Delta^* \cdot A_{21} \cdot A_{11}^* & \Delta^* \end{bmatrix}$$

(with $\Delta = A_{22} + A_{21} \cdot A_{11}^* \cdot A_{12}$)

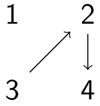
with proof that it satisfies $w^* \simeq 1 + w \cdot w^*$

Reachability example

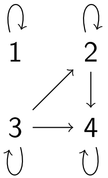


$$\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}^*$$

Reachability example



$$\begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{bmatrix}^* = \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$



Wrapping up

Conclusions, further work, et.c.

- This matrix definition is useable...
- A more flexible matrix definition: sparse? fewer constructors?
- Automation (of proofs)!
- Generalisation to closed semi-near-ring for parsing applications.
- Agda development available at `https://github.com/DSLsofMath/FLABloM`.