On the Decidability of Conversion in Type Theory

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Introduction

We present a proof of the following theorem

Theorem

Let $\Gamma \vdash t : A$ and $\Gamma \vdash u : A$. It is decidable whether $\Gamma \vdash t = u : A$.

- Proof by Kripke logical relations defined on weak head reduction.
- Two logical relations.
  - Π injectivity.
  - Conversion.

Syntax

- Language

$$t, u, A, B := x \mid U \mid N \mid \Pi(x : A)B \mid \lambda x.t \mid t\ u \mid$$
$$0 \mid \mathsf{S}\ t \mid \mathsf{natrec}\ (\lambda x.A)\ t\ u$$

- Neutral terms

$$k := x \mid \mathsf{S}\ k \mid k\ u \mid \mathsf{natrec}\ (\lambda x.A)\ c_z\ g\ k$$

Typed Weak Head Reduction

- One step weak head reduction

$$\frac{\Gamma \vdash t \to t' : \Pi(x{:}A)B \qquad \Gamma \vdash a : A}{\Gamma \vdash t\, a \to t' a : B[a]} \qquad \frac{\Gamma, x{:}A \vdash t : B \qquad \Gamma \vdash a : A}{\Gamma \vdash (\lambda x.t)\, a \to t[a/x] : B[a]}$$

$$\frac{\Gamma \vdash t \to u : A \qquad \Gamma \vdash A = B}{\Gamma \vdash t \to u : B} \qquad \frac{\Gamma \vdash A \to B : \mathsf{U}}{\Gamma \vdash A \to B}$$

- Define $\to^*$ the reflexive transitive closure of $\to$

1st Logical Relation

$$\frac{\Gamma \vdash A \to^* K \qquad K \text{ neutral}}{\Gamma \Vdash A}$$

- $\Gamma \Vdash A = B$ if

  $\Gamma \vdash B \to^* L$ with L neutral and
  $\Gamma \vdash K = L$.
- $\Gamma \Vdash t : A$ if $\Gamma \vdash t \to^* l : A$ with l neutral.
- $\Gamma \Vdash t = u : A$ if

  $\Gamma \vdash t \to^* l : A$ with l neutral and
  $\Gamma \vdash u \to^* k : A$ with k neutral and
  $\Gamma \vdash l = k : A$.

1st Logical Relation

$$\frac{\Gamma \vdash A \to^* \Pi(x{:}F)G \qquad \Gamma \Vdash F \\ (\forall \Delta \leq \Gamma)(\Delta \Vdash a : F \Rightarrow \Delta \Vdash G[a]) \\ (\forall \Delta \leq \Gamma)(\Delta \Vdash a = b : F \Rightarrow \Delta \Vdash G[a] = G[b])}{\Gamma \Vdash A}$$

- $\Gamma \Vdash A = B$ if
  $\Gamma \Vdash B$ with $\Gamma \vdash B \to^* \Pi(x{:}H)E$ and
  $\Gamma \Vdash F = H$ and
  for all $\Delta \leq \Gamma$, $(\Delta \Vdash a : F) \Rightarrow (\Delta \Vdash G[a] = E[a])$

1st Logical Relation

- $\Gamma \Vdash f : A$ if $\Gamma \vdash f : A$ and for all $\Delta \leq \Gamma$,
  $(\Delta \Vdash a : F) \Rightarrow (\Delta \Vdash f\ a : G[a])$ and
  $(\Delta \Vdash a = b : F) \Rightarrow (\Delta \Vdash f\ a = f\ b : G[a])$
- $\Gamma \Vdash f = g : A$ if
  $\Gamma \Vdash f : A$ and $\Gamma \Vdash g : A$ and
  for all $\Delta \leq \Gamma$, $(\Delta \Vdash a : F) \Rightarrow (\Delta \Vdash f\ a = g\ a : G[a])$

1st Logical Relation

- Straightforward from the definition

Lemma If $\Gamma \Vdash J$ then $\Gamma \vdash J$.

- Completeness (By induction on the type system)

Lemma If $\Gamma \vdash J$ then $\Gamma \Vdash J$

- Corollaries
  - If $\Gamma \vdash \Pi(x{:}F)G = B$ then $\Gamma \vdash B \to^* \Pi(x{:}H)E$ and $\Gamma \vdash F = H$ and $\Gamma, x{:}F \vdash G = E$.
  - If $\Gamma \vdash \lambda x.t : \Pi(x{:}F)G$ then $\Gamma, x{:}F \vdash t : G$.
  - If t is neutral and $\Gamma \vdash t : A$ and $\Gamma \vdash t : B$ then $\Gamma \vdash A = B$.
  - If $\vdash t : N$ then $\vdash t \to^* \mathsf{S}^m 0 : N$.

Conversion

$$\frac{x : A \in \Gamma}{\Gamma \vdash x \sim x}$$

$$\frac{\Gamma \vdash k \sim l \quad \Gamma \vdash k : \Pi(x : F)G \quad \Gamma \vdash t \text{ conv } v : F}{\Gamma \vdash k\,t \sim l\,v}$$

$$\frac{\Gamma \vdash k \sim l \quad \Gamma \vdash A \to^* M \quad \Gamma \vdash k : A}{\Gamma \vdash k \text{ conv } l : A}\ M \text{ neutral or base type}$$

Conversion

$$\frac{\Gamma \vdash F \,\mathsf{conv}\, H \quad \Gamma, x : F \vdash G[x] \,\mathsf{conv}\, E[x]}{\Gamma \vdash \Pi(x : F)G \,\mathsf{conv}\, \Pi(x : H)E}$$

$$\frac{\begin{array}{l}\Gamma \vdash f : \Pi(x : F)G \\ \Gamma \vdash g : \Pi(x : F)G \quad \Gamma, x : F \vdash f\,x \;\mathsf{conv}\; g\,x : G[x]\end{array}}{\Gamma \vdash f \;\mathsf{conv}\; g : \Pi(x : F)G}$$

Conversion

$$\frac{\Gamma \vdash A \to^* A' \quad \Gamma \vdash B \to^* B' \quad \Gamma \vdash A' \,\mathrm{conv}\, B'}{\Gamma \vdash A \,\mathrm{conv}\, B}$$

$$\frac{\Gamma \vdash A \to^* B \quad \Gamma \vdash a \to^* a' : B \quad \Gamma \vdash b \to^* b' : B \quad \Gamma \vdash a' \,\mathrm{conv}\, b' : B}{\Gamma \vdash a \,\mathrm{conv}\, b : A}$$

Lemma

Let $\Gamma \vdash t : A$ and $\Gamma \vdash u : A$. It is decidable whether $\Gamma \vdash t \,\mathit{conv}\, u : A$.

Straightforward from the definition we have

- If $\Gamma \vdash a \,\mathrm{conv}\, b : A$ then $\Gamma \vdash a = b : A$

2nd Logical Relation

Except for neutral types and terms the logical relation remain the same.

$$\frac{\Gamma \vdash A \to^* K \qquad K \text{ neutral} \qquad \Gamma \vdash K \operatorname{conv} K}{\Gamma \Vdash A}$$

- $\Gamma \Vdash A = B$ if $A \operatorname{conv} B$.
- $\Gamma \Vdash t : A$ if $\Gamma \vdash t \operatorname{conv} t : A$.
- $\Gamma \Vdash t = u : A$ if

  $\Gamma \Vdash t : A$ and $\Gamma \Vdash u : A$ and

  $\Gamma \Vdash t \operatorname{conv} u : A$.

2nd Logical Relation

- Easy direction

Lemma If $\Gamma \Vdash a = b : A$ then $\Gamma \vdash a \operatorname{conv} b : A$.

- Completeness (By induction on the type system)

Lemma If $\Gamma \vdash J$ then $\Gamma \Vdash J$.

Theorem
If $\Gamma \vdash a : A$ and $\Gamma \vdash b : A$ then $\Gamma \vdash a = b : A$ iff $\Gamma \vdash a \operatorname{conv} b : A$.
That is, it is decidable whether $\Gamma \vdash a = b : A$.