Components of a Hammer for Type Theory

Goal Translation and Proof Reconstruction

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Interactive Proof in Type Theory

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- Why do we hate it?

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  - Rewriting: Simp, Subst, HORewrite
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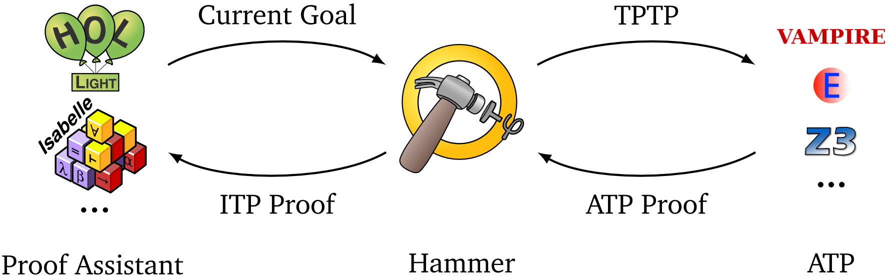
  - Decision Procedures: Congruence Closure, Ring, Omega, Cooper, ...
- AI/ATP techniques: Hammers
  - MizAR for Mizar
  - Sledgehammer for Isabelle/HOL
  - HOL(y)Hammer for HOL Light and HOL4

Hammer Overview



Evaluations

Top-level goals:

- HOL(y)Hammer
  - Flyspeck text formalization: 47%
  - Similar results for HOL4 and CakeML
- Sledgehammer
  - Probability theory: 40%
  - Term rewriting: 44%
  - Java threads: 59%
- MizAR
  - Mizar Mathematical Library: 40%

More for subgoals

For Type Theory?

Premise selection

- Features
- Machine Learning

Encoding CoC and variants in formalisms of ATPs

- Soundness? Completeness? Efficiency!
- This talk

Reconstruction: Get an ITP proof

- Extract information from the ATP proof
- Redo the proof

Translation

Target logic

Target logic: untyped FOL with equality.

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Three functions $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{C}$.

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  - $\mathscr{C}_\Gamma(\lambda \vec{x} : \vec{t}.s) = F\vec{y}$ where s does not start with a lambda-abstraction any more, F is a fresh constant, $\vec{y} = \mathrm{FV}(\lambda \vec{x} : \vec{t}.s)$ and $\forall \vec{y}.\mathscr{F}_\Gamma(\forall \vec{x} : \vec{t}.F\vec{y}\vec{x} = s)$ is a new axiom.

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- Translate induction principles and recursor definitions.

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- Do automatic proof search using our tactic `yreconstr`.

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- If a subterm of the form match x with . . . occurs in the goal or in one of the hypotheses, then destruct x.
- Use an `isolve` tactic at the leaves of the search tree: a combination of Coq's `congruence`, `subst`, `easy`, `eauto` tactics, some hypotheses simplification and goal splitting.

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Prover	Solved%	Solved	Sum%	Sum	Unique
Vampire	32.9	6839	32.9	6839	855
Z3	27.6	5734	34.9	7265	390
E Prover	25.8	5376	35.3	7337	72
any	35.3	7337	35.3	7337	

Table 1: Results of the experimental evaluation on the 20803 FOL problems generated from the propositions in the Coq standard library.

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  - about 70% provable using `firstorder isolve` provided that generic equality axioms are added to the context.

Experimental evaluation

Tactic	Time	Solved%	Solved
`yreconstr`	1s	83.1	6097
`yreconstr`	2s	85.8	6296
`yreconstr`	5s	87.5	6421
`yreconstr`	10s	88.1	6466
`yreconstr`	15s	88.2	6473
`simple`	1s	50.1	3674
`firstorder'`	10s	69.6	5103
`jprover`	10s	56.1	4114
any		90.1	6609

Table 2: Results of the evaluation of proof reconstruction on the 7337 problems solved by the ATPs.

Conclusion

- Provided missing components of a hammer for type theory
- Efficient encoding in FOL
  - Able to automatically prove 35% of Coq's standard library
- Simple reconstruction
  - 90% of the ATP-found proofs can be rebuilt in Coq
- Other libraries?
  - Mathematical Components / SS-Reflect where different automation?
  - Libraries of Matita, Lean, ...?
- Optimize, optimize, optimize!
  - Learning
  - Translation
  - Reconstruction