A Type Theory for Comprehensive Parametric Polymorphism

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Joint work with...

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Parametric polymorphism [Strachey, 1967]

A polymorphic program

$$t : \forall \alpha.\, A$$

is parametric if it applies the same uniform algorithm at all instantiations $t[B]$ of its type parameter.

Typical example:

$$\mathtt{reverse} : \forall \alpha.\, \mathtt{List}\ \alpha \to \mathtt{List}\ \alpha$$

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A polymorphic program $t : \forall\alpha. A$ is relationally parametric if for all relations $R \subseteq B \times B'$,

$$(t[B], t[B']) \in \langle A \rangle(R)$$

where $\langle A \rangle(R) \subseteq A(B) \times A(B')$ is the relational interpretation of the type A.

E.g. `reverse` $: \forall\alpha. \texttt{List}\ \alpha \to \texttt{List}\ \alpha$ is relationally parametric.

Applications of relational parametricity

Relational parametricity enables:

- Reasoning about abstract data types.
- Correctness (universal properties) of encodings of data types.
- 'Theorems for free!' [Wadler, 1989].

Usually in the setting of $\lambda\mathbf{2}$ (System F) [Girard, 1972; Reynolds, 1974] — serves as a model type theory for (impredicative) polymorphism.

How can we reason about $\lambda\mathbf{2}$ terms using relational parametricity?

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Backed up by a semantic framework, i.e. the type theory is the 'internal language' of a class of models of $\lambda\mathbf{2}$ (comprehensive $\lambda\mathbf{2}$ parametricity graphs) [Ghani, N. F., and Simpson, 2016].

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Expected properties of parametricity can be proved using the type theory, but the proof involves novel ingredients due to minimality of structure:

- definability of direct image relations,
- arguments without use of equality relations, and
- only weak forms of graph relations available ('pseudographs').

Extending $\lambda\mathbf{2}$ with relations: the type theory $\lambda\mathbf{2R}$

Judgement forms of $\lambda\mathbf{2}$:

Γ ctxt	Γ is a context
$\Gamma \vdash A$ type	A is a type in context Γ
$\Gamma \vdash t : A$	term t has type A in context Γ
$\Gamma \vdash t = s : A$	judgemental equality

Note: Single context with term and type variables interleaved — motivated by semantics.

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$\Theta \vdash A_1 R A_2$ rel	R is a relation between types A_1 and A_2
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Not conservative extension of $\lambda\mathbf{2}$ — point is to derive stronger properties.

Matthew 6:3

Importantly, in a judgement

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the "left hand side" $(\Theta)_1 \vdash t_1 : A_1$ and the "right hand side" $(\Theta)_2 \vdash t_2 : A_2$ are treated completely separately.

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Imposed by the semantics, with practical consequences: cannot talk about equality relations.

Forgetting relations

Relational judgements contain left and right ordinary judgements:

$$
\begin{aligned}
(\cdot)_i &= \cdot \\
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Lemma.

$$
\begin{aligned}
\Theta \ \mathsf{rctxt} &\implies (\Theta)_i \ \mathsf{ctxt} \\
\Theta \vdash A_1 R A_2 \ \mathsf{rel} &\implies (\Theta)_i \vdash A_i \ \mathsf{type} \\
\Theta \vdash (t_1 \mathbin{:} A_1) R (t_2 \mathbin{:} A_2) &\implies (\Theta)_i \vdash t_i \mathbin{:} A_i
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Inverse image relations

$$\frac{\Theta \vdash B_1 R B_2 \text{ rel} \qquad (\Theta)_1 \vdash t_1 : A_1 \to B_1 \qquad (\Theta)_2 \vdash t_2 : A_2 \to B_2}{\Theta \vdash A_1([t_1 \times t_2]^{-1} R) A_2 \text{ rel}}$$

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$$\frac{\Theta \vdash (t_1\, u_1 : B_1) R (t_2\, u_2 : B_2)}{\Theta \vdash (u_1 : A_1)([t_1 \times t_2]^{-1} R)(u_2 : A_2)}$$

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$$[t_1 \times t_2]_! R := [i_{B_1} \times i_{B_2}]^{-1}(\forall \alpha \rho \beta. ([(- \circ t_1) \times (- \circ t_2)]^{-1}(R \to \rho)) \to \rho)$$

where i_B abbreviates $\lambda b. \Lambda \alpha. \lambda t. t\, b : B \to \forall \alpha. (B \to \alpha) \to \alpha$.

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Theorem. *Every comprehensive $\lambda\mathbf{2}$ parametricity graph contains a family of fibrewise opfibrations.*

This will also be important for proving the expected properties of parametricity.

Doubling up

Have already seen left and right projections $(\cdot)_1$, $(\cdot)_2$.

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Conversely, a "doubling" operation $\langle\cdot\rangle$ takes typing contexts to relational contexts.

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Lemma (Reynold's Abstraction Theorem).

$$\Gamma \vdash t : A \implies \langle\Gamma\rangle \vdash (t : A)\langle A\rangle(t : A)$$

Relatedness rules

$$\frac{}{\Theta \vdash (x_1 : A_1) R (x_2 : A_2)} \; ((x_1 : A_1) R (x_2 : A_2) \in \Theta)$$

$$\frac{\Theta, (x_1 : A_1) R (x_2 : A_2) \vdash (t_1 : B_1) S (t_2 : B_2)}{\Theta \vdash (\lambda x_1 . \, t_1 : A_1 \to B_1)(R \to S)(\lambda x_2 . \, t_2 : A_2 \to B_2)}$$

$$\frac{\Theta \vdash (s_1 : A_1 \to B_1)(R \to S)(s_2 : A_2 \to B_2) \qquad \Theta \vdash (t_1 : A_1) R (t_2 : A_2)}{\Theta \vdash (s_1 \, t_1 : B_1) S (s_2 \, t_2 : B_2)}$$

$$\frac{\Theta, \alpha \rho \beta \vdash (t_1 : A_1) R (t_2 : A_2)}{\Theta \vdash (\Lambda \alpha . \, t_1 : \forall \alpha . \, A_1)\big(\forall \alpha \rho \beta . \, R\big)(\Lambda \beta . \, t_2 : \forall \beta . \, A_2)} \qquad \frac{\langle \Gamma \rangle \vdash (s : A) \langle A \rangle (t : A)}{\Gamma \vdash s = t : A}$$

$$\frac{\Theta \vdash (t_1 : \forall \alpha . \, A_1)\big(\forall \alpha \rho \beta . \, R\big)(t_2 : \forall \beta . \, A_2) \qquad \Theta \vdash B_1 S B_2 \text{ rel}}{\Theta \vdash (t_1[B_1] : A_1[\alpha \mapsto B_1]) R[\alpha \rho \beta \mapsto B_1 S B_2](t_2[B_2] : A_2[\beta \mapsto B_2])}$$

$$\frac{\Theta \vdash (t_1 : A_1) R (t_2 : A_2) \qquad \Theta_1 \vdash t_1 = s_1 : A_1 \qquad \Theta_2 \vdash t_2 = s_2 : A_2}{\Theta \vdash (s_1 : A_1) R (s_2 : A_2)}$$

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$$\text{True} \implies \langle \Gamma \rangle \vdash (t : \forall \alpha.\, B)\big(\forall \rho.\, \langle B \rangle\big)(t : \forall \alpha.\, B)$$

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Subtlety required:

- Graph relations usually key for arguments.

This

Note … nges.

In fac

Typic

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8. *Terms of type* $\forall\alpha.\,F(\alpha,\alpha) \to G(\alpha,\alpha)$ *for mixed-variance type expressions* F *and* G *are dinatural.*

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4. $\forall\alpha.\,(A\to\alpha)\to(B\to\alpha)\to\alpha$ *is* $A+B$.
5. $\forall\alpha.\,(\forall\beta.\,(T(\beta) \to \alpha)) \to \alpha$ *is* $\exists\alpha.\,T(\alpha)$.
6. *The type* $\forall\alpha.\,(T(\alpha) \to \alpha) \to \alpha$ *is the carrier of the initial T-algebra for all functorial type expressions* $T(\alpha)$.
7. *The type* $\exists\alpha.\,(\alpha \to T(\alpha)) \times \alpha$ *is the carrier of the final T-coalgebra for all functorial type expressions* $T(\alpha)$.
8. *Terms of type* $\forall\alpha.\,F(\alpha,\alpha) \to G(\alpha,\alpha)$ *for mixed-variance type expressions F and G are dinatural.*

Initial algebras use inverse image pseudographs, final coalgebras direct image ones.

Summary

- A type theory $\lambda\mathbf{2R}$ for reasoning about relational parametricity for System F.

- Sound and complete semantics in comprehensive $\lambda\mathbf{2}$ parametricity graphs.

- Proof of consequences of parametricity using the type theory involves novel ingredients:
  - direct image relations via impredicative encoding,
  - no identity relations available, and
  - two different pseudo-graph relations (using inverse and direct images).
- **Future work:** Extend to e.g. dependent type theory.

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Thank you!

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Semantic framework

Definition (Comprehensive λ2 parametricity graph). *A comprehensive λ**2** parametricity graph is a reflexive graph of comprehensive λ**2** fibrations*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{R}(\mathbb{T}) \arrow[r, shift left=2, "\nabla_1^{\mathbb{T}},\ \Delta^{\mathbb{T}},\ \nabla_2^{\mathbb{T}}"] \arrow[r, shift right=2] \arrow[d, "p^{\mathcal{R}}"'] & \mathbb{T} \arrow[l] \arrow[d, "p"] \\
\mathcal{R}(\mathbb{C}) \arrow[r, shift left=2] \arrow[r, shift right=2, "\nabla_1^{\mathbb{C}},\ \Delta^{\mathbb{C}},\ \nabla_2^{\mathbb{C}}"'] & \mathbb{C} \arrow[l]
\end{tikzcd}$$

which is "fibrewise" a parametricity graph.

λ2 fibrations [Seely, 1987; see also Jacobs, 1999]

Definition (λ2 fibration). A $\lambda\mathbf{2}$ fibration is a split fibration $p : \mathbb{T} \to \mathbb{C}$, where the base category $\mathbb{C}$ has finite products, and the fibration:

1. is fibred cartesian closed;
2. has a split generic object U — we write Ω for $p\,U$;
3. and has fibred-products along projections $X \times \Omega \longrightarrow X$ in $\mathbb{C}$.

Moreover, the reindexing functors given by the splitting should preserve the above-specified structure in fibres on the nose.

Definition (Comprehensive λ2 fibration). *A $\lambda\mathbf{2}$ fibration $p : \mathbb{T} \to \mathbb{C}$ is comprehensive if it enjoys the comprehension property: the fibred-terminal-object functor $X \mapsto \mathbf{1}_X : \mathbb{C} \to \mathbb{T}$ has a specified right adjoint $K : \mathbb{T} \to \mathbb{C}$.*

Definition (Fibrewise parametricity graph).

A reflexive graph of (comprehensive) $\lambda\mathbf{2}$ fibrations

$$\begin{tikzcd}
\mathcal{R}(\mathbb{T}) \arrow[r, shift left=2, "\nabla_1^{\mathbb{T}},\ \Delta^{\mathbb{T}},\ \nabla_2^{\mathbb{T}}"] \arrow[r, shift right=2] \arrow[d, "p^{\mathcal{R}}"'] & \mathbb{T} \arrow[l] \arrow[d, "p"] \\
\mathcal{R}(\mathbb{C}) \arrow[r, shift left=2] \arrow[r, shift right=2, "\nabla_1^{\mathbb{C}},\ \Delta^{\mathbb{C}},\ \nabla_2^{\mathbb{C}}"'] & \mathbb{C} \arrow[l]
\end{tikzcd}$$

is fibrewise a parametricity graph if for all $W \in \mathcal{R}(\mathbb{C})$ and $X \in \mathbb{C}$:

(Relational) $\langle \nabla_1^{\mathbb{T}}, \nabla_2^{\mathbb{T}} \rangle \restriction_{\mathcal{R}(\mathbb{T})_W} : \mathcal{R}(\mathbb{T})_W \to \mathbb{T}_{\nabla_1^{\mathbb{C}} W} \times \mathbb{T}_{\nabla_2^{\mathbb{C}} W}$ is faithful.

(Identity property) $\Delta^{\mathbb{T}} \restriction_{\mathbb{T}_X} : \mathbb{T}_X \to \mathcal{R}(\mathbb{T})_{\Delta^{\mathbb{C}} X}$ is full.

(Fibration) $\langle \nabla_1^{\mathbb{T}}, \nabla_2^{\mathbb{T}} \rangle \restriction_{\mathcal{R}(\mathbb{T})_W} : \mathcal{R}(\mathbb{T})_W \to \mathbb{T}_{\nabla_1^{\mathbb{C}} W} \times \mathbb{T}_{\nabla_2^{\mathbb{C}} W}$ is a fibration.

Moreover, the fibration $\langle \nabla_1^{\mathbb{T}}, \nabla_2^{\mathbb{T}} \rangle \restriction_{\mathcal{R}(\mathbb{T})_W}$ should be cloven, and reindexing should give rise to a cleavage-preserving fibred functor from $\langle \nabla_1^{\mathbb{T}}, \nabla_2^{\mathbb{T}} \rangle \restriction_{\mathcal{R}(\mathbb{T})_W}$ to $\langle \nabla_1^{\mathbb{T}}, \nabla_2^{\mathbb{T}} \rangle \restriction_{\mathcal{R}(\mathbb{T})_{W'}}$.