



# If-then-else and other constructive and classical connectives

(Or: How to derive natural deduction rules from truth tables)

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# Outline

Natural Deduction and Truth Tables

Kripke models

Cut-elimination and Curry-Howard





# Truth tables

Classically, the meaning of a propositional connective is fixed by its **truth table**. This immediately implies

- consistency,
- a decision procedure,
- completeness (w.r.t. Boolean algebra's).

Intuitionistically, the meaning of a connective is fixed by explaining what a **proof** is that involves the connective.

Basically, this explains the **introduction rule** for the connective.

By analysing proofs we can then also get

- consistency (from proof normalization and analysing normal deductions),
- a decision procedure (from the subformula property for normal deductions),
- completeness (w.r.t. Heyting algebra's).



# Natural Deduction rules from truth tables

Let  $c$  be an  $n$ -ary connective  $c$  with truth table  $t_c$  and write  $\Phi = c(A_1, \dots, A_n)$ .

Each row of  $t_c$  gives rise to an elimination rule or an introduction rule for  $c$ .

$\begin{array}{c c} A_1 & \dots & A_n &   & \Phi \\ \hline a_1 & \dots & a_n &   & 0 \end{array}$	$\mapsto$	$\frac{\vdash \Phi \dots \vdash A_j \text{ (if } a_j = 1) \dots A_i \vdash D \text{ (if } a_i = 0) \dots}{\vdash D} \text{el}$
$\begin{array}{c c} A_1 & \dots & A_n &   & \Phi \\ \hline b_1 & \dots & b_n &   & 1 \end{array}$	$\mapsto$	$\frac{\dots \vdash A_j \text{ (if } b_j = 1) \dots A_i \vdash \Phi \text{ (if } b_i = 0) \dots}{\vdash \Phi} \text{in}^i$
$\begin{array}{c c} A_1 & \dots & A_n &   & \Phi \\ \hline b_1 & \dots & b_n &   & 1 \end{array}$	$\mapsto$	$\frac{\Phi \vdash D \dots \vdash A_j \text{ (if } b_j = 1) \dots A_i \vdash D \text{ (if } b_i = 0) \dots}{\vdash D} \text{in}^c$



# Examples

Intuitionistic rules for  $\wedge$  (3 elim rules and one intro rule):

$A$	$B$	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

$$\frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D} \wedge\text{-el}_a$$

$$\frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D} \wedge\text{-el}_b$$

$$\frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D} \wedge\text{-el}_c$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \wedge\text{-in}$$

- These rules can be shown to be equivalent to the well-known intuitionistic rules.
- These rules can be optimized to 3 rules.



# Examples

Rules for  $\neg$ : 1 elimination rule and 1 introduction rule.

$A$	$\neg A$
0	1
1	0

Intuitionistic:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D} \neg\text{-el} \qquad \frac{A \vdash \neg A}{\vdash \neg A} \neg\text{-in}^i$$

Classical:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D} \neg\text{-el} \qquad \frac{\neg A \vdash D \quad A \vdash D}{\vdash D} \neg\text{-in}^c$$





# Lemma 1 to simplify the rules

$$\frac{\begin{array}{c} \vdash \phi_1 \dots \vdash \phi_n \quad \psi_1 \vdash D \dots \psi_m \vdash D \quad A \vdash D \\ \hline \vdash D \\ \hline \vdash \phi_1 \dots \vdash \phi_n \quad \vdash A \quad \psi_1 \vdash D \dots \psi_m \vdash D \\ \hline \vdash D \end{array}}{\vdash D}$$

is equivalent to the system with these two rules replaced by

$$\frac{\vdash \phi_1 \dots \vdash \phi_n \quad \psi_1 \vdash D \dots \psi_m \vdash D}{\vdash D}$$



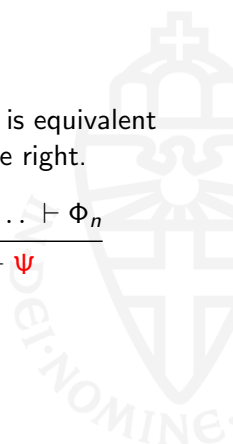


# Lemma II to simplify the rules

A system with a deduction rule of the form to the left is equivalent to the system with this rule replaced by the rule on the right.

$$\frac{\vdash \Phi_1 \dots \vdash \Phi_n \quad \Psi \vdash D}{\vdash D}$$

$$\frac{\vdash \Phi_1 \dots \vdash \Phi_n}{\vdash \Psi}$$







# The intuitionistic connectives

We have already seen the  $\wedge, \neg$  rules. The optimised rules for  $\vee, \rightarrow, \top$  and  $\perp$  we obtain are:

$$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D} \vee\text{-el}$$

$$\frac{\vdash A}{\vdash A \vee B} \vee\text{-in}_1$$

$$\frac{\vdash B}{\vdash A \vee B} \vee\text{-in}_2$$

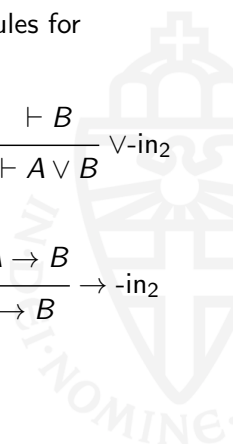
$$\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B} \rightarrow\text{-el}$$

$$\frac{\vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_1$$

$$\frac{A \vdash A \rightarrow B}{\vdash A \rightarrow B} \rightarrow\text{-in}_2$$

$$\frac{}{\vdash \top} \top\text{-in}$$

$$\frac{\vdash \perp}{\vdash D} \perp\text{-el}$$





# The rules for the classical $\rightarrow$ connective

Deduction of Peirce's law:

$$\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B} \rightarrow\text{-el}$$

$$\frac{\vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_1$$

$$\frac{A \vdash D \quad A \rightarrow B \vdash D}{\vdash D} \rightarrow\text{-in}_2^C$$

$$(A \rightarrow B) \rightarrow A \vdash (A \rightarrow B) \rightarrow A \quad A \rightarrow B \vdash A \rightarrow B$$

$$A \rightarrow B, (A \rightarrow B) \rightarrow A \vdash A$$

$$A \rightarrow B, (A \rightarrow B) \rightarrow A \vdash ((A \rightarrow B) \rightarrow A) \rightarrow A$$

$$\frac{A \vdash A}{A \vdash ((A \rightarrow B) \rightarrow A) \rightarrow A}$$

$$A \rightarrow B \vdash ((A \rightarrow B) \rightarrow A) \rightarrow A$$

$$\frac{A \vdash ((A \rightarrow B) \rightarrow A) \rightarrow A \quad A \rightarrow B \vdash ((A \rightarrow B) \rightarrow A) \rightarrow A}{\vdash ((A \rightarrow B) \rightarrow A) \rightarrow A} \rightarrow\text{-in}_2^C$$



# The “If Then Else” connective

Notation:  $A \rightarrow B / C$  for if  $A$  then  $B$  else  $C$ .

$p$	$q$	$r$	$p \rightarrow q / r$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

The optimized intuitionistic rules are:

$$\frac{\vdash A \rightarrow B / C \quad \vdash A}{\vdash B} \text{ then-el}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \rightarrow B / C} \text{ then-in}$$

$$\frac{\vdash A \rightarrow B / C \quad A \vdash D \quad C \vdash D}{\vdash D} \text{ else-el}$$

$$\frac{A \vdash A \rightarrow B / C \quad \vdash C}{\vdash A \rightarrow B / C} \text{ else-in}$$





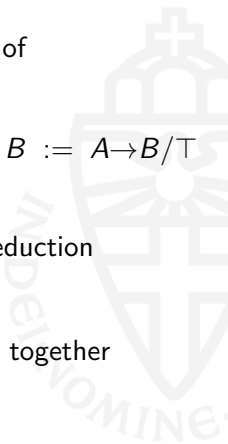
# “If Then Else” $\dot{+}$ $\dot{\top}$ $\dot{\perp}$ is functionally complete

We define the usual intuitionistic connectives in terms of if-then-else,  $\dot{\top}$  and  $\dot{\perp}$ :

$$A \dot{\vee} B := A \rightarrow A/B \quad A \dot{\wedge} B := A \rightarrow B/A \quad A \dot{\rightarrow} B := A \rightarrow B/\dot{\top}$$

LEMMA The defined connectives satisfy the original deduction rules for these same connectives.

COROLLARY The intuitionistic connective if-then-else, together with  $\dot{\top}$  and  $\dot{\perp}$ , is functionally complete.





## Kripke semantics for the intuitionistic rules

For each  $n$ -ary connective  $c$ , we assume a truth table  $t_c : \{0, 1\}^n \rightarrow \{0, 1\}$  and the defined deduction rules.

DEFINITION A **Kripke model** is a triple  $(W, \leq, \text{at})$  where  $W$  is a set of worlds,  $\leq$  a reflexive, transitive relation on  $W$  and a function  $\text{at} : W \rightarrow \wp(\text{At})$  satisfying  $w \leq w' \Rightarrow \text{at}(w) \subseteq \text{at}(w')$ .

We define the notion  **$\varphi$  is true in world  $w$**  (usually written  $w \Vdash \varphi$ ) by defining  $\llbracket \varphi \rrbracket_w \in \{0, 1\}$

DEFINITION of  $\llbracket \varphi \rrbracket_w \in \{0, 1\}$ , by induction on  $\varphi$ :

- (atom) if  $\varphi$  is atomic,  $\llbracket \varphi \rrbracket_w = 1$  iff  $\varphi \in \text{at}(w)$ .
- (connective) for  $\varphi = c(\varphi_1, \dots, \varphi_n)$ ,  $\llbracket \varphi \rrbracket_w = 1$  iff for each  $w' \geq w$ ,  $t_c(\llbracket \varphi_1 \rrbracket_{w'}, \dots, \llbracket \varphi_n \rrbracket_{w'}) = 1$  where  $t_c$  is the truth table of  $c$ .

$\Gamma \Vdash \psi :=$  for each Kripke model and each world  $w$ , if  $\llbracket \varphi \rrbracket_w = 1$  for each  $\varphi$  in  $\Gamma$ , then  $\llbracket \psi \rrbracket_w = 1$ .



## Kripke semantics for the intuitionistic rules

THEOREM  $\Gamma \vdash \varphi$  if and only if  $\Gamma \models \varphi$

Proof. Soundness ( $\Rightarrow$ ) is by induction on the deduction of  $\Gamma \vdash \varphi$ .

For completeness we need to construct a special Kripke model that “corresponds to the deduction system”.

- In the literature, the completeness of Kripke semantics is proved using *prime theories*.
- A theory is prime if it satisfies the **disjunction property**: if  $\Gamma \vdash A \vee B$ , then  $\Gamma \vdash A$  or  $\Gamma \vdash B$ .
- We may not have  $\vee$  in our set of connective, and we may have others that “behave  $\vee$ -like”.
- So we need to pass by the disjunction property.
- (But we can generalize the disjunction property to arbitrary  $n$ -ary intuitionistic connectives.)
- We consider pairs  $(\Gamma, \psi)$  where  $\Gamma$  is a  **$\psi$ -maximal theory**, a maximal theory that does **not prove**  $\psi$ .



## A generalised disjunction property

We say that the  $n$ -ary connective  $c$  is  *$i, j$ -splitting* in case the truth table for  $c$  has the following shape

$p_1$	...	$p_i$	...	$p_j$	...	$p_n$	$c(p_1, \dots, p_n)$
—	...	0	...	0	...	—	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
—	...	0	...	0	...	—	0

In terms of  $t_c$ :

$$t_c(p_1, \dots, p_{i-1}, 0, p_{i+1}, \dots, p_{j-1}, 0, p_{j+1}, \dots, p_n) = 0$$

for all  $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n \in \{0, 1\}$ .

LEMMA Let  $c$  be an  $i, j$ -splitting connective and suppose  $\vdash c(A_1, \dots, A_n)$ . Then  $\vdash A_i$  or  $\vdash A_j$ .



# Examples of connectives with a splitting property

$p$	$q$	$r$	$\text{most}(p, q, r)$	$p \rightarrow q/r$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

- $\text{most}$  is  $i, j$ -splitting for every  $i, j$ :
  - if  $\vdash \text{most}(A_1, A_2, A_3)$ , then  $\vdash A_i$  or  $\vdash A_j$ , for any pair  $i \neq j$ .
- if-then-else is 1, 3-splitting and 2, 3-splitting (but not 1, 2-splitting):
  - if  $\vdash A \rightarrow B/C$ , then  $\vdash A$  or  $\vdash C$  and also  $\vdash B$  or  $\vdash C$ .
  - if  $\vdash A \rightarrow B/C$ , then **not**  $\vdash A$  or  $\vdash B$







# Substituting a deduction in another

LEMMA: If  $\Gamma \vdash \varphi$  and  $\Delta, \varphi \vdash \psi$ , then  $\Gamma, \Delta \vdash \psi$

If  $\Sigma$  is a deduction of  $\Gamma \vdash \varphi$  and  $\Pi$  is a deduction of  $\Delta, \varphi \vdash \psi$ , then we have the following deduction of  $\Gamma, \Delta \vdash \psi$ :

$$\begin{array}{c}
 \vdots \Sigma \quad \quad \quad \vdots \Sigma \\
 \vdots \quad \quad \quad \vdots \\
 \Gamma \vdash \varphi \quad \dots \quad \Gamma \vdash \varphi \\
 \vdots \quad \quad \quad \vdots \Pi \\
 \vdots \quad \quad \quad \vdots \\
 \Delta \vdash \psi
 \end{array}$$

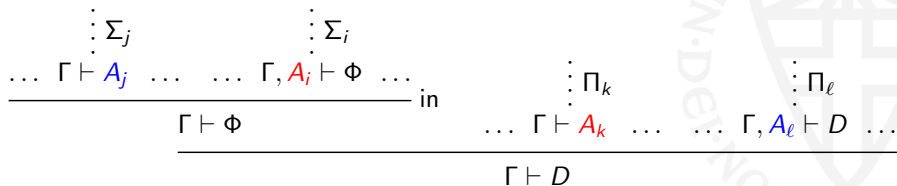
In  $\Pi$ , every application of an (axiom) rule at a leaf, deriving  $\Delta' \vdash \varphi$  for some  $\Delta' \supseteq \Delta$  is replaced by a copy of a deduction  $\Sigma$ , which is also a deduction of  $\Delta', \Gamma \vdash \varphi$ .



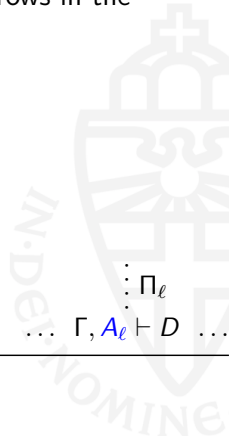
# Cuts in intuitionistic logic

An **intuitionistic direct cut** is a pattern of the following form, where  $\Phi = c(A_1, \dots, A_n)$ . Remember these rules arise from rows in the truth table  $t_c$ :

$p_1$	$\dots$	$p_n$	$c(p_1, \dots, p_n)$
$a_1$	$\dots$	$a_n$	0
$b_1$	$\dots$	$b_n$	1



- $b_j = 1$  for  $A_j$  and  $b_i = 0$  for  $A_i$
- $a_k = 1$  for  $A_k$  and  $a_\ell = 0$  for  $A_\ell$





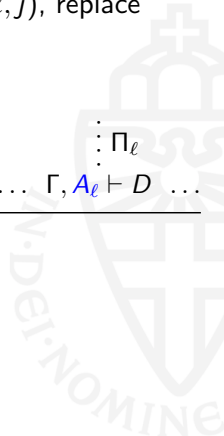
# Eliminating a direct cut (I)

The *elimination of a direct cut* is defined by replacing the deduction pattern by another one. If  $\ell = j$  (for some  $\ell, j$ ), replace

$$\frac{
 \begin{array}{c}
 \vdots \Sigma_j \\
 \dots \Gamma \vdash A_j \dots
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \Sigma_i \\
 \dots \Gamma, A_i \vdash \Phi \dots
 \end{array}
 }{
 \Gamma \vdash \Phi
 }
 \quad
 \begin{array}{c}
 \vdots \Pi_k \\
 \dots \Gamma \vdash A_k \dots
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \Pi_\ell \\
 \dots \Gamma, A_\ell \vdash D \dots
 \end{array}
 }{
 \Gamma \vdash D
 }$$

by

$$\begin{array}{c}
 \vdots \Sigma_j \quad \vdots \Sigma_j \\
 \Gamma \vdash A_j \quad \dots \quad \Gamma \vdash A_j \\
 \vdots \Pi_\ell \\
 \Gamma \vdash D
 \end{array}$$





## Eliminating a direct cut (II)

If  $k = i$  (for some  $k, i$ ), replace

$$\frac{
 \begin{array}{c}
 \vdots \Sigma_j \\
 \dots \Gamma \vdash A_j \dots
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \Sigma_i \\
 \dots \Gamma, A_i \vdash \Phi \dots
 \end{array}
 }{
 \Gamma \vdash \Phi
 }
 \quad
 \frac{
 \begin{array}{c}
 \vdots \Pi_k \\
 \dots \Gamma \vdash A_k \dots
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \Pi_\ell \\
 \dots \Gamma, A_\ell \vdash D \dots
 \end{array}
 }{
 \Gamma \vdash D
 }$$

by

$$\frac{
 \begin{array}{c}
 \vdots \Pi_k \\
 \Gamma \vdash A_i
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \Pi_k \\
 \dots \Gamma \vdash A_i
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \Sigma_i \\
 \Gamma \vdash \Phi
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \Pi_k \\
 \dots \Gamma \vdash A_i \dots
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \Pi_\ell \\
 \dots \Gamma, A_\ell \vdash D \dots
 \end{array}
 }{
 \Gamma \vdash D
 }$$



# Cuts for if-then-else (I)

The cut-elimination rules for if-then-else are the following.

(then-then)

$$\frac{\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \rightarrow B/C} \text{ in} \quad \Gamma \vdash A}{\Gamma \vdash B} \text{ el}$$

$\vdots \Sigma$   
 $\vdots$

$\mapsto$

$$\frac{\vdots \Sigma}{\Gamma \vdash B}$$

(else-then)

$$\frac{\frac{\Gamma, A \vdash A \rightarrow B/C \quad \Gamma \vdash C}{\Gamma \vdash A \rightarrow B/C} \text{ in} \quad \Gamma \vdash A}{\Gamma \vdash B} \text{ el}$$

$\vdots \Sigma$   
 $\vdots \Pi$

$\mapsto$

$$\frac{\frac{\Gamma \vdash A \quad \Gamma \vdash A}{\Gamma \vdash A \rightarrow B/C} \text{ in} \quad \Gamma \vdash A}{\Gamma \vdash B} \text{ el}$$

$\vdots \Pi$      $\vdots \Pi$   
 $\vdots \Sigma$      $\vdots \Pi$





# Cuts for if-then-else (II)

(then-else)

$$\frac{\frac{\begin{array}{c} \vdots \\ \Sigma \end{array} \quad \Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \rightarrow B/C} \text{ in} \quad \frac{\begin{array}{c} \vdots \\ \Pi \end{array} \quad \Gamma, A \vdash D \quad \Gamma, C \vdash D}{\Gamma \vdash D} \text{ el}}{\Gamma \vdash D}$$

$\mapsto$

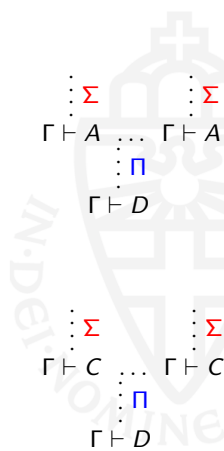
$$\frac{\begin{array}{c} \vdots \\ \Sigma \end{array} \quad \Gamma \vdash A \quad \dots \quad \Gamma \vdash A \quad \begin{array}{c} \vdots \\ \Sigma \end{array} \quad \frac{\begin{array}{c} \vdots \\ \Pi \end{array} \quad \Gamma \vdash D}{\Gamma \vdash D} \text{ el}}{\Gamma \vdash D}$$

(else-else)

$$\frac{\frac{\Gamma, A \vdash A \rightarrow B/C \quad \begin{array}{c} \vdots \\ \Sigma \end{array} \quad \Gamma \vdash C}{\Gamma \vdash A \rightarrow B/C} \text{ in} \quad \frac{\begin{array}{c} \vdots \\ \Pi \end{array} \quad \Gamma, A \vdash D \quad \Gamma, C \vdash D}{\Gamma \vdash D} \text{ el}}{\Gamma \vdash D}$$

$\mapsto$

$$\frac{\begin{array}{c} \vdots \\ \Sigma \end{array} \quad \Gamma \vdash C \quad \dots \quad \Gamma \vdash C \quad \begin{array}{c} \vdots \\ \Sigma \end{array} \quad \frac{\begin{array}{c} \vdots \\ \Pi \end{array} \quad \Gamma \vdash D}{\Gamma \vdash D} \text{ el}}{\Gamma \vdash D}$$





# Curry-Howard proofs-as-terms

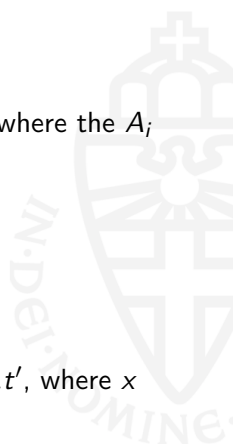
We define rules for the judgment  $\Gamma \vdash t : A$ , where

- $A$  is a formula,
- $\Gamma$  is a set of declarations  $\{x_1 : A_1, \dots, x_m : A_m\}$ , where the  $A_i$  are formulas and the  $x_i$  are term-variables,
- $t$  is a **proof-term**.

For a connective  $c \in \mathcal{C}$  of arity  $n$ , we have

- an **introduction term**  $\iota(t_1, \dots, t_n)$ ,
- an **elimination term**  $\varepsilon(t_0, t_1, \dots, t_n)$ ,

where the  $t_i$  are again proof-terms or of the shape  $\lambda x.t'$ , where  $x$  is a term-variable and  $t'$  is a proof-term.





# The calculus $\lambda$ if-then-else

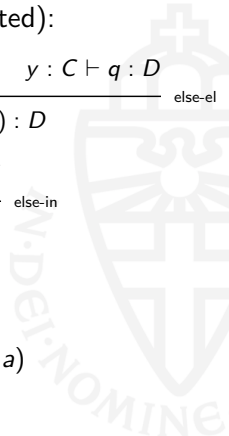
DEFINITION We define the calculus  $\lambda$ if-then-else as a calculus for terms and reductions for the if-then-else logic ( $\Gamma$  omitted):

$$\frac{\vdash t_0 : A \rightarrow B / C \quad \vdash a : A}{\vdash \varepsilon_1(t_0, a) : B} \text{ then-el} \qquad \frac{\vdash t_0 : A \rightarrow B / C \quad x : A \vdash t : D \quad y : C \vdash q : D}{\vdash \varepsilon_2(t_0, \lambda x.t, \lambda y.q) : D} \text{ else-el}$$

$$\frac{\vdash a : A \quad \vdash b : B}{\vdash \iota_1(a, b) : A \rightarrow B / C} \text{ then-in} \qquad \frac{x : A \vdash t : A \rightarrow B / C \quad \vdash c : C}{\vdash \iota_2(\lambda x.t, c) : A \rightarrow B / C} \text{ else-in}$$

The reduction rules are

$$\begin{aligned} \varepsilon_1(\iota_1(a, b), a') &\longrightarrow b \\ \varepsilon_1(\iota_2(\lambda x.t, c), a) &\longrightarrow \varepsilon_1(t[x := a], a) \\ \varepsilon_2(\iota_1(a, b), \lambda x.t, \lambda y.q) &\longrightarrow t[x := a] \\ \varepsilon_2(\iota_2(\lambda x.t, c), \lambda z.d, \lambda y.q) &\longrightarrow q[y := c] \end{aligned}$$







# Strong Normalization for $\lambda$ if-then-else

We prove Strong Normalization for the reductions in  $\lambda$ if-then-else by adapting the saturated sets method.

But ... what we would **really** want is that proof-terms in **normal form** have the **subformula property**: if  $t : A$ , then the type of a sub-term of  $t$  is a sub-type of  $A$ .

Then we can derive

- consistency of the logic
- decidability of the logic
- and thereby a (simple?) decision procedure for full IPC.

We need to add **permuting reduction rules**

$$\begin{aligned} \epsilon_1(\epsilon_2(t_0, \lambda x.t, \lambda y.q), e) &\longrightarrow \epsilon_2(t_0, \lambda x.\epsilon_1(t, e), \lambda y.\epsilon_1(q, e)) \\ \epsilon_2(\epsilon_2(t_0, \lambda x.t, \lambda y.q), \lambda v.r, \lambda z.s) &\longrightarrow \epsilon_2(t_0, \lambda x.\epsilon_2(t, \lambda v.r, \lambda z.s), \lambda y.\epsilon_2(q, \lambda v.r, \lambda z.s)) \end{aligned}$$



## Conclusions, Further work, Related work

### Conclusions

- Simple way to construct deduction system for new connectives, intuitionistically and classically
- Study connectives “in isolation”. (Without defining them.)
- Generic Kripke semantics

### Some open questions/ further work:

- Constructive proof of the completeness of Kripke semantics
- Meaning of the new connectives as data types
- General definition of classical cut-elimination
- Relation with other term calculi for classical logic: subtraction logic,  $\lambda\mu$  (Parigot),  $\bar{\lambda}\mu\tilde{\mu}$  (Curien, Herbelin).
- SN for  $\lambda$ if-then-else with permuting cuts

### Related work:

- Jan von Plato and Sara Negri
- Peter Schroeder-Heister