If-then-else and other constructive and classical connectives (Or: How to derive natural deduction rules from truth tables)

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Outline

Truth tables

Classically, the meaning of a propositional connective is fixed by its truth table. This immediately implies

- consistency,
- a decision procedure,
- completeness (w.r.t. Boolean algebra's).

Intuitionistically, the meaning of a connective is fixed by explaining what a proof is that involves the connective.
Basically, this explains the introduction rule for the connective.
By analysing proofs we can then also get

- consistency (from proof normalization and analysing normal deductions),
- a decision procedure (from the subformula property for normal deductions),
- completeness (w.r.t. Heyting algebra's).

Natural Deduction rules from truth tables

Let c be an n-ary connective c with truth table t_c and write $\Phi = c(A_1, \ldots, A_n)$.
Each row of t_c gives rise to an elimination rule or an introduction rule for c.

A_1	$\ldots$	A_n	Φ
a_1	$\ldots$	a_n	0

$$\mapsto \quad \frac{\vdash \Phi \ldots \vdash A_j \text{ (if } a_j = 1) \ldots A_i \vdash D \text{ (if } a_i = 0) \ldots}{\vdash D}\,\text{el}$$

A_1	$\ldots$	A_n	Φ
b_1	$\ldots$	b_n	1

$$\mapsto \quad \frac{\ldots \vdash A_j \text{ (if } b_j = 1) \ldots A_i \vdash \Phi \text{ (if } b_i = 0) \ldots}{\vdash \Phi}\,\text{in}^i$$

A_1	$\ldots$	A_n	Φ
b_1	$\ldots$	b_n	1

$$\mapsto \quad \frac{\Phi \vdash D \ldots \vdash A_j \text{ (if } b_j = 1) \ldots A_i \vdash D \text{ (if } b_i = 0) \ldots}{\vdash D}\,\text{in}^c$$

Examples

Intuitionistic rules for $\wedge$ (3 elim rules and one intro rule):

A	B	$A \wedge B$
0	0	0
0	1	0
1	0	0
1	1	1

$$\frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D}\ \wedge\text{-el}_a \qquad \frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D}\ \wedge\text{-el}_b$$

$$\frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D}\ \wedge\text{-el}_c \qquad \frac{\vdash A \quad \vdash B}{\vdash A \wedge B}\ \wedge\text{-in}$$

- These rules can be shown to be equivalent to the well-known intuitionistic rules.
- These rules can be optimized to 3 rules.

Examples

Rules for $\neg$: 1 elimination rule and 1 introduction rule.

A	$\neg A$
0	1
1	0

Intuitionistic:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D}\ \neg\text{-el} \qquad \frac{A \vdash \neg A}{\vdash \neg A}\ \neg\text{-in}^i$$

Classical:

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D}\ \neg\text{-el} \qquad \frac{\neg A \vdash D \quad A \vdash D}{\vdash D}\ \neg\text{-in}^c$$

Lemma I to simplify the rules

$$\frac{\vdash \Phi_1 \ \ldots \ \vdash \Phi_n \quad \Psi_1 \vdash D \ \ldots \ \Psi_m \vdash D \quad A \vdash D}{\vdash D}$$

$$\frac{\vdash \Phi_1 \ \ldots \ \vdash \Phi_n \quad \vdash A \quad \Psi_1 \vdash D \ \ldots \ \Psi_m \vdash D}{\vdash D}$$

is equivalent to the system with these two rules replaced by

$$\frac{\vdash \Phi_1 \ \ldots \ \vdash \Phi_n \quad \Psi_1 \vdash D \ \ldots \ \Psi_m \vdash D}{\vdash D}$$

Lemma II to simplify the rules

A system with a deduction rule of the form to the left is equivalent to the system with this rule replaced by the rule on the right.

$$\frac{\vdash \Phi_1 \quad \ldots \quad \vdash \Phi_n \qquad \Psi \vdash D}{\vdash D} \qquad\qquad \frac{\vdash \Phi_1 \quad \ldots \quad \vdash \Phi_n}{\vdash \Psi}$$

The intuitionistic connectives

We have already seen the $\wedge, \neg$ rules. The optimised rules for $\vee, \rightarrow, \top$ and $\bot$ we obtain are:

$$\frac{\vdash A \vee B \quad A \vdash D \quad B \vdash D}{\vdash D}\ \vee\text{-el} \qquad \frac{\vdash A}{\vdash A \vee B}\ \vee\text{-in}_1 \qquad \frac{\vdash B}{\vdash A \vee B}\ \vee\text{-in}_2$$

$$\frac{\vdash A \rightarrow B \quad \vdash A}{\vdash B} \rightarrow\text{-el} \qquad \frac{\vdash B}{\vdash A \rightarrow B} \rightarrow\text{-in}_1 \qquad \frac{A \vdash A \rightarrow B}{\vdash A \rightarrow B} \rightarrow\text{-in}_2$$

$$\frac{}{\vdash \top}\ \top\text{-in} \qquad \frac{\vdash \bot}{\vdash D}\ \bot\text{-el}$$

The rules for the classical $\to$ connective

Deduction of Peirce's law:

$$\frac{\vdash A \to B \quad \vdash A}{\vdash B} \to\text{-el} \qquad \frac{\vdash B}{\vdash A \to B} \to\text{-in}_1 \qquad \frac{A \vdash D \quad A \to B \vdash D}{\vdash D} \to\text{-in}_2^c$$

$$\dfrac{\dfrac{A \vdash A}{A \vdash ((A \to B) \to A) \to A} \quad \dfrac{\dfrac{\dfrac{(A \to B) \to A \vdash (A \to B) \to A \quad A \to B \vdash A \to B}{A \to B, (A \to B) \to A \vdash A}}{A \to B, (A \to B) \to A \vdash ((A \to B) \to A) \to A}}{A \to B \vdash ((A \to B) \to A) \to A}}{\vdash ((A \to B) \to A) \to A} \to\text{-in}_2^c$$

The "If Then Else" connective

Notation: $A \to B/C$ for if A then B else C.

p	q	r	$p \to q/r$
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

The optimized intuitionistic rules are:

$$\frac{\vdash A \to B/C \quad \vdash A}{\vdash B}\ \text{then-el} \qquad \frac{\vdash A \to B/C \quad A \vdash D \quad C \vdash D}{\vdash D}\ \text{else-el}$$

$$\frac{\vdash A \quad \vdash B}{\vdash A \to B/C}\ \text{then-in} \qquad \frac{A \vdash A \to B/C \quad \vdash C}{\vdash A \to B/C}\ \text{else-in}$$

"If Then Else" $+\top+\bot$ is functionally complete

We define the usual intuitionistic connectives in terms of if-then-else, $\top$ and $\bot$:

$$A \dot{\vee} B := A{\to}A/B \qquad A \dot{\wedge} B := A{\to}B/A \qquad A \dot{\to} B := A{\to}B/\top$$

LEMMA The defined connectives satisfy the original deduction rules for these same connectives.

COROLLARY The intuitionistic connective if-then-else, together with $\top$ and $\bot$, is functionally complete.

Kripke semantics for the intuitionistic rules

For each n-ary connective c, we assume a truth table $t_c : \{0,1\}^n \to \{0,1\}$ and the defined deduction rules.

DEFINITION A Kripke model is a triple $(W, \leq, \mathrm{at})$ where W is a set of worlds, $\leq$ a reflexive, transitive relation on W and a function $\mathrm{at} : W \to \wp(\mathrm{At})$ satisfying $w \leq w' \Rightarrow \mathrm{at}(w) \subseteq \mathrm{at}(w')$.

We define the notion φ is true in world w (usually written $w \Vdash \varphi$) by defining $[\![\varphi]\!]_w \in \{0,1\}$

DEFINITION of $[\![\varphi]\!]_w \in \{0,1\}$, by induction on φ:

- (atom) if φ is atomic, $[\![\varphi]\!]_w = 1$ iff $\varphi \in \mathrm{at}(w)$.
- (connective) for $\varphi = c(\varphi_1, \ldots, \varphi_n)$, $[\![\varphi]\!]_w = 1$ iff for each $w' \geq w$, $t_c([\![\varphi_1]\!]_{w'}, \ldots, [\![\varphi_n]\!]_{w'}) = 1$ where t_c is the truth table of c.

$\Gamma \models \psi$:= for each Kripke model and each world w, if $[\![\varphi]\!]_w = 1$ for each φ in Γ, then $[\![\psi]\!]_w = 1$.

Kripke semantics for the intuitionistic rules

THEOREM $\Gamma \vdash \varphi$ if and only if $\Gamma \models \varphi$

Proof. Soundness ($\Rightarrow$) is by induction on the deduction of $\Gamma \vdash \varphi$.

For completeness we need to construct a special Kripke model that "corresponds to the deduction system".

- In the literature, the completeness of Kripke semantics is proved using *prime theories*.
- A theory is prime if it satisfies the disjunction property: if $\Gamma \vdash A \vee B$, then $\Gamma \vdash A$ or $\Gamma \vdash B$.
- We may not have $\vee$ in our set of connective, and we may have others that "behave $\vee$-like"',
- So we need to pass by the disjunction property.
- (But we can generalize the disjunction property to arbitrary *n*-ary intuitionistic connectives.)
- We consider pairs (Γ, ψ) where Γ is a ψ-maximal theory, a maximal theory that does not prove ψ.

A generalised disjunction property

We say that the n-ary connective c is i, j-splitting in case the truth table for c has the following shape

p_1	$\dots$	p_i	$\dots$	p_j	$\dots$	p_n	$c(p_1, \dots, p_n)$
$-$	$\dots$	0	$\dots$	0	$\dots$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$-$	$\dots$	0	$\dots$	0	$\dots$	$-$	0

In terms of t_c:

$$t_c(p_1, \dots, p_{i-1}, 0, p_{i+1}, \dots, p_{j-1}, 0, p_{j+1}, \dots, p_n) = 0$$

for all $p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_{j-1}, p_{j+1}, \dots, p_n \in \{0, 1\}$.

LEMMA Let c be an i, j-splitting connective and suppose $\vdash c(A_1, \dots, A_n)$. Then $\vdash A_i$ or $\vdash A_j$.

Examples of connectives with a splitting property

p	q	r	most(p, q, r)	$p \to q/r$
0	0	0	0	0
0	0	1	0	1
0	1	0	0	0
0	1	1	1	1
1	0	0	0	0
1	0	1	1	0
1	1	0	1	1
1	1	1	1	1

- most is i, j-splitting for every i, j:
  - if $\vdash$ most(A_1, A_2, A_3), then $\vdash A_i$ or $\vdash A_j$, for any pair $i \neq j$.
- if-then-else is $1, 3$-splitting and $2, 3$-splitting (but not $1, 2$-splitting):
  - if $\vdash A \to B/C$, then $\vdash A$ or $\vdash C$ and also $\vdash B$ or $\vdash C$.
  - if $\vdash A \to B/C$, then not $\vdash A$ or $\vdash B$

Substituting a deduction in another

LEMMA: If $\Gamma \vdash \varphi$ and $\Delta, \varphi \vdash \psi$, then $\Gamma, \Delta \vdash \psi$

If Σ is a deduction of $\Gamma \vdash \varphi$ and Π is a deduction of $\Delta, \varphi \vdash \psi$, then we have the following deduction of $\Gamma, \Delta \vdash \psi$:

$$
\begin{array}{ccc}
\vdots\ \Sigma & & \vdots\ \Sigma \\
\Gamma \vdash \varphi & \ldots & \Gamma \vdash \varphi \\
& \vdots\ \Pi & \\
& \Delta \vdash \psi &
\end{array}
$$

In Π, every application of an (axiom) rule at a leaf, deriving $\Delta' \vdash \varphi$ for some $\Delta' \supseteq \Delta$ is replaced by a copy of a deduction Σ, which is also a deduction of $\Delta', \Gamma \vdash \varphi$.

Cuts in intuitionistic logic

An intuitionistic direct cut is a pattern of the following form, where $\Phi = c(A_1, \ldots, A_n)$. Remember these rules arise from rows in the truth table t_c:

p_1	$\ldots$	p_n	$c(p_1, \ldots, p_n)$
a_1	$\ldots$	a_n	0
b_1	$\ldots$	b_n	1

$$\dfrac{\dfrac{\ldots\quad \begin{array}{c}\vdots\ \Sigma_j\\ \Gamma \vdash A_j\end{array} \quad \ldots \qquad \ldots \quad \begin{array}{c}\vdots\ \Sigma_i\\ \Gamma, A_i \vdash \Phi\end{array} \quad \ldots}{\Gamma \vdash \Phi} \text{ in } \qquad \ldots \quad \begin{array}{c}\vdots\ \Pi_k\\ \Gamma \vdash A_k\end{array} \quad \ldots \qquad \ldots \quad \begin{array}{c}\vdots\ \Pi_\ell\\ \Gamma, A_\ell \vdash D\end{array} \quad \ldots}{\Gamma \vdash D}$$

- $b_j = 1$ for A_j and $b_i = 0$ for A_i
- $a_k = 1$ for A_k and $a_\ell = 0$ for A_ℓ

Eliminating a direct cut (I)

The *elimination of a direct cut* is defined by replacing the deduction pattern by another one. If $\ell = j$ (for some ℓ, j), replace

$$\dfrac{\dfrac{\ldots \quad \begin{matrix}\vdots\ \Sigma_j \\ \Gamma \vdash A_j\end{matrix} \quad \ldots \qquad \ldots \quad \begin{matrix}\vdots\ \Sigma_i \\ \Gamma, A_i \vdash \Phi\end{matrix} \quad \ldots}{\Gamma \vdash \Phi} \qquad \ldots \quad \begin{matrix}\vdots\ \Pi_k \\ \Gamma \vdash A_k\end{matrix} \quad \ldots \qquad \ldots \quad \begin{matrix}\vdots\ \Pi_\ell \\ \Gamma, A_\ell \vdash D\end{matrix} \quad \ldots}{\Gamma \vdash D}$$

by

$$\begin{matrix} \begin{matrix}\vdots\ \Sigma_j \\ \Gamma \vdash A_j\end{matrix} \quad \ldots \quad \begin{matrix}\vdots\ \Sigma_j \\ \Gamma \vdash A_j\end{matrix} \\ \vdots\ \Pi_\ell \\ \Gamma \vdash D \end{matrix}$$

Eliminating a direct cut (II)

If $k = i$ (for some k, i), replace

$$\dfrac{\dfrac{\ldots\ \ \begin{array}{c}\vdots\ \Sigma_j\\ \Gamma \vdash A_j\end{array}\ \ \ldots \qquad \ldots\ \ \begin{array}{c}\vdots\ \Sigma_i\\ \Gamma, {\color{red}A_i} \vdash \Phi\end{array}\ \ \ldots}{\Gamma \vdash \Phi} \qquad \ldots\ \ \begin{array}{c}\vdots\ \Pi_k\\ \Gamma \vdash {\color{red}A_k}\end{array}\ \ \ldots \qquad \ldots\ \ \begin{array}{c}\vdots\ \Pi_\ell\\ \Gamma, A_\ell \vdash D\end{array}\ \ \ldots}{\Gamma \vdash D}$$

by

$$\dfrac{\begin{array}{c}\begin{array}{c}\vdots\ \Pi_k\\ \Gamma \vdash A_i\end{array}\ \ \ldots\ \ \begin{array}{c}\vdots\ \Pi_k\\ \Gamma \vdash A_i\end{array}\\ \vdots\ \Sigma_i\\ \Gamma \vdash \Phi\end{array} \qquad \ldots\ \ \begin{array}{c}\vdots\ \Pi_k\\ \Gamma \vdash A_i\end{array}\ \ \ldots \qquad \ldots\ \ \begin{array}{c}\vdots\ \Pi_\ell\\ \Gamma, A_\ell \vdash D\end{array}\ \ \ldots}{\Gamma \vdash D}$$

Cuts for if-then-else (I)

The cut-elimination rules for if-then-else are the following.

(then-then)

$$\dfrac{\dfrac{\Gamma \vdash A \qquad \begin{array}{c}\vdots\ \Sigma\\ \Gamma \vdash B\end{array}}{\Gamma \vdash A \to B/C}\,\text{in} \qquad \Gamma \vdash A}{\Gamma \vdash B}\,\text{el} \quad \mapsto \quad \begin{array}{c}\vdots\ \Sigma\\ \Gamma \vdash B\end{array}$$

(else-then)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \Sigma\\ \Gamma, A \vdash A \to B/C\end{array} \qquad \Gamma \vdash C}{\Gamma \vdash A \to B/C}\,\text{in} \qquad \begin{array}{c}\vdots\ \Pi\\ \Gamma \vdash A\end{array}}{\Gamma \vdash B}\,\text{el} \quad \mapsto \quad \dfrac{\begin{array}{c}\begin{array}{c}\vdots\ \Pi\\ \Gamma \vdash A\end{array} \ \ldots \ \begin{array}{c}\vdots\ \Pi\\ \Gamma \vdash A\end{array}\\ \vdots\ \Sigma\\ \Gamma \vdash A \to B/C\end{array} \qquad \begin{array}{c}\vdots\ \Pi\\ \Gamma \vdash A\end{array}}{\Gamma \vdash B}\,\text{el}$$

Cuts for if-then-else (II)

(then-else)

$$\dfrac{\dfrac{\begin{array}{c}\vdots\,\Sigma\\ \Gamma\vdash A\end{array}\quad \Gamma\vdash B}{\Gamma\vdash A{\to}B/C}\,\text{in} \qquad \begin{array}{c}\vdots\,\Pi\\ \Gamma, A\vdash D\end{array} \qquad \Gamma, C\vdash D}{\Gamma\vdash D}\,\text{el} \quad\mapsto\quad \begin{array}{c}\begin{array}{c}\vdots\,\Sigma\\ \Gamma\vdash A\end{array}\ \ldots\ \begin{array}{c}\vdots\,\Sigma\\ \Gamma\vdash A\end{array}\\ \vdots\,\Pi\\ \Gamma\vdash D\end{array}$$

(else-else)

$$\dfrac{\dfrac{\Gamma, A\vdash A{\to}B/C \quad \begin{array}{c}\vdots\,\Sigma\\ \Gamma\vdash C\end{array}}{\Gamma\vdash A{\to}B/C}\,\text{in} \qquad \Gamma, A\vdash D \qquad \begin{array}{c}\vdots\,\Pi\\ \Gamma, C\vdash D\end{array}}{\Gamma\vdash D}\,\text{el} \quad\mapsto\quad \begin{array}{c}\begin{array}{c}\vdots\,\Sigma\\ \Gamma\vdash C\end{array}\ \ldots\ \begin{array}{c}\vdots\,\Sigma\\ \Gamma\vdash C\end{array}\\ \vdots\,\Pi\\ \Gamma\vdash D\end{array}$$

Curry-Howard proofs-as-terms

We define rules for the judgment $\Gamma \vdash t : A$, where

- A is a formula,
- Γ is a set of declarations $\{x_1 : A_1, \ldots, x_m : A_m\}$, where the A_i are formulas and the x_i are term-variables,
- t is a proof-term.

For a connective $c \in \mathcal{C}$ of arity n, we have

- an introduction term $\iota(t_1, \ldots, t_n)$,
- an elimination term $\varepsilon(t_0, t_1, \ldots, t_n)$,

where the t_i are again proof-terms or of the shape $\lambda x.t'$, where x is a term-variable and t' is a proof-term.

The calculus λif-then-else

DEFINITION We define the calculus λif-then-else as a calculus for terms and reductions for the if-then-else logic (Γ omitted):

$$\frac{\vdash t_0 : A \to B/C \quad \vdash a : A}{\vdash \varepsilon_1(t_0, a) : B}\ \text{then-el} \qquad \frac{\vdash t_0 : A \to B/C \quad x : A \vdash t : D \quad y : C \vdash q : D}{\vdash \varepsilon_2(t_0, \lambda x.t, \lambda y.q) : D}\ \text{else-el}$$

$$\frac{\vdash a : A \quad \vdash b : B}{\vdash \iota_1(a, b) : A \to B/C}\ \text{then-in} \qquad \frac{x : A \vdash t : A \to B/C \quad \vdash c : C}{\vdash \iota_2(\lambda x.t, c) : A \to B/C}\ \text{else-in}$$

The reduction rules are

$$\begin{array}{lcl}
\varepsilon_1(\iota_1(a, b), a') & \longrightarrow & b \\
\varepsilon_1(\iota_2(\lambda x.t, c), a) & \longrightarrow & \varepsilon_1(t[x := a], a) \\
\varepsilon_2(\iota_1(a, b), \lambda x.t, \lambda y.q) & \longrightarrow & t[x := a] \\
\varepsilon_2(\iota_2(\lambda x.t, c), \lambda z.d, \lambda y.q) & \longrightarrow & q[y := c]
\end{array}$$

Strong Normalization for λif-then-else

We prove Strong Normalization for the reductions in λif-then-else by adapting the saturated sets method.

But . . . what we would really want is that proof-terms in normal form have the subformula property: if $t : A$, then the type of a sub-term of t is a sub-type of A.

Then we can derive

- consistency of the logic
- decidability of the logic
- and thereby a (simple?) decision procedure for full IPC.

We need to add permuting reduction rules

$$\begin{array}{rcl} \varepsilon_1(\varepsilon_2(t_0, \lambda x.t, \lambda y.q), e) & \longrightarrow & \varepsilon_2(t_0, \lambda x.\varepsilon_1(t, e), \lambda y.\varepsilon_1(q, e)) \\ \varepsilon_2(\varepsilon_2(t_0, \lambda x.t, \lambda y.q), \lambda v.r, \lambda z.s) & \longrightarrow & \varepsilon_2(t_0, \lambda x.\varepsilon_2(t, \lambda v.r, \lambda z.s), \lambda y.\varepsilon_2(q, \lambda v.r, \lambda z.s)) \end{array}$$

Conclusions, Further work, Related work

Conclusions

- Simple way to construct deduction system for new connectives, intuitionistically and classically
- Study connectives "in isolation". (Without defining them.)
- Generic Kripke semantics

Some open questions/ further work:

- Constructive proof of the completeness of Kripke semantics
- Meaning of the new connectives as data types
- General definition of classical cut-elimination
- Relation with other term calculi for classical logic: subtraction logic, $\lambda\mu$ (Parigot), $\bar{\lambda}\mu\tilde{\mu}$ (Curien, Herbelin).
- SN for λif-then-else with permuting cuts

Related work:

- Jan von Plato and Sara Negri
- Peter Schroeder-Heister