β reduction without rule ξ

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Overview

- A concrete representation of lambda terms.
- Locally nameless:
    - indexes for bound positions,
    - names for free variables.
    - Canonical: α conversion is syntactic identity.
- Abstraction, $\text{lam}_x M$, is a defined function.
- Using the defined abstraction, the language looks like conventional notation.
- We can define various reduction relations without rule ξ.
- Only works for some relations.
    - Apparently fails for η.

Preterms and well formedness

We use names for free variables, and indexes for bound variables.

- Let i, j, m, n, p, q, range over natural numbers.
- Fix a countable set of *names*, ranged over by x, y, z.
- The raw syntax of *preterms* (ranged over by M, N, P, Q) is

$$\text{pt} ::= \; \mathsf{X}_n x \; \mid \; \mathsf{J}_n j \; \mid \; \lceil M, N \rceil_n$$

In preterm syntax, n is the *height* of the preterm, written *hgt M*.

Well formedness (written $\mathcal{W}M$) is defined inductively by

$$\frac{}{\mathcal{W}\mathsf{X}_n x} \qquad \frac{j < n}{\mathcal{W}\mathsf{J}_n j} \qquad \frac{\mathcal{W}P \quad \mathcal{W}Q \quad n \le \text{hgt } P \quad n \le \text{hgt } Q}{\mathcal{W}\lceil P, Q \rceil_n}$$

- Well formed terms are called *terms*.

Intended Meaning of (Well Formed) Terms

$$\frac{}{\mathcal{W} \mathsf{X}_n\, x} \qquad \frac{j < n}{\mathcal{W} \mathsf{J}_n\, j} \qquad \frac{\mathcal{W} P \quad \mathcal{W} Q \quad n \le hgt\, P \quad n \le hgt\, Q}{\mathcal{W} \lceil P,\, Q \rceil_n}$$

- $\mathsf{X}_n\, x$ represents $\lambda_1 \ldots \lambda_n\, x$.
- $\mathsf{J}_n\, j$ represents $\lambda_1 \ldots \lambda_n\, j$.
  - Require $j < n$ for well formedness; otherwise j would be unbound.
- Consider $\mathcal{W} P$ (representing $\lambda_1 \ldots \lambda_p\, T_P$)
     and $\mathcal{W} Q$ (representing $\lambda_1 \ldots \lambda_q\, T_Q$).
  - By induction, every index in P or Q is bound.
  - $\lceil P,\, Q \rceil_0$ represents $((\lambda_1 \ldots \lambda_p\, T_P)\ (\lambda_1 \ldots \lambda_q\, T_Q))$.
  - $\lceil P,\, Q \rceil_1$ represents $\lambda_1((\lambda_2 \ldots \lambda_p\, T_P)\ (\lambda_2 \ldots \lambda_q\, T_Q))$.
  - $\lceil P,\, Q \rceil_2$ represents $\lambda_1\, \lambda_2((\lambda_3 \ldots \lambda_p\, T_P)\ (\lambda_3 \ldots \lambda_q\, T_Q))$.
  - . . .
  - Thus well formedness of $\lceil P,\, Q \rceil_n$ requires
    $n \le \min(hgt\, P,\, hgt\, Q)$.

Abstraction defined as a function on preterms

$$\text{lam}_x(X_n \, y) := \text{if } x = y \text{ then } J_{n+1} \, 0 \text{ else } X_{n+1} \, y$$
$$\text{lam}_x(J_n \, j) := J_{n+1} \, (j+1)$$
$$\text{lam}_x \lceil M, N \rceil_n := \lceil \text{lam}_x M, \text{lam}_x N \rceil_{n+1}$$

- Abstraction preserves well formedness and raises height by one.

$$\mathcal{W}M \implies \mathcal{W}(\text{lam}_x M) \qquad hgt\,(\text{lam}_x M) = hgt\,M + 1$$

- Conversely, every term with height a successor is an abstraction.

$$\mathcal{W}M \wedge hgt\,M = n + 1 \implies \exists\,P\,x\,.\,M = \text{lam}_x P$$

- We use A, B as metavariables over abstractions.

Examples

Notations: write

- $\mathsf{lam}_{xy}M$ for $\mathsf{lam}_x\mathsf{lam}_yM$.
- $\overline{x}$ for $\mathsf{X}_0\,x$.

Some combinators: (assuming $x \neq y$, $x \neq z$, $y \neq z$)

$$\begin{aligned}
I &= \lambda x \,.\, x = \mathsf{lam}_x\overline{x} = \mathsf{J}_1\,0 \\
K &= \lambda x\,y \,.\, x = \mathsf{lam}_{xy}\overline{x} = \mathsf{J}_2\,0 \\
\mathit{false} &= \lambda x\,y \,.\, y = \mathsf{lam}_{xy}\overline{y} = \mathsf{J}_2\,1
\end{aligned}$$

$$\begin{aligned}
S &= \lambda x\,y\,z \,.\, (x\,z)\,(y\,z) \\
&= \mathsf{lam}_{xyz}\lceil\lceil\overline{x},\overline{z}\rceil_0, \lceil\overline{y},\overline{z}\rceil_0\rceil_0 \\
&= \lceil\lceil\mathsf{J}_3\,0, \mathsf{J}_3\,2\rceil_3, \lceil\mathsf{J}_3\,1, \mathsf{J}_3\,2\rceil_3\rceil_3
\end{aligned}$$

The isomorphism between terms and informal lambda terms:

$$x \sim \mathsf{X}_0\,x \qquad \frac{t_1 \sim M_1 \quad t_2 \sim M_2}{(t_1\,t_2) \sim \lceil M_1, M_2\rceil_0} \qquad \frac{t \sim M}{\lambda x.t \sim \mathsf{lam}_x M}$$

Lifting

To define instantiation we first introduce a lifting function

$$
(X_n\, y)^{\uparrow} := X_{n+1}\, y
$$
$$
(J_n\, j)^{\uparrow} := J_{n+1}\, (j+1)
$$
$$
(\lceil M,\, N \rceil_n)^{\uparrow} := \lceil (M)^{\uparrow},\, (N)^{\uparrow} \rceil_{n+1}
$$

which we iterate as:

$$
(M)^{\uparrow 0} := M
$$
$$
(M)^{\uparrow m+1} := ((M)^{\uparrow m})^{\uparrow}
$$

- Lifting preserves well formedness and raises height by one.

$$
\mathcal{W} M \implies \mathcal{W}(M)^{\uparrow} \wedge hgt\,(M)^{\uparrow} = hgt\, M + 1
$$

Instantiation

Instantiation is a binary function, $M[N]$.

- If $hgt\ M = 0$ (M is under no binders), $M[N] = M$.
- Otherwise $M[N]$ fills any holes $J_{n+1}\ 0$ in M and adjusts the rest of the term:

$$X_{n+1}\ y[N] := X_n\ y$$
$$J_{n+1}\ 0[N] := (N)^{\uparrow n}$$
$$J_{n+1}\ (j{+}1)[N] := J_n\ j$$
$$\lceil M, P \rceil_{n+1}[N] := \lceil M[N], P[N] \rceil_n$$

- Instantiation is not substitution.
- Instantiation preserves well formedness:

$$\mathcal{W}M \wedge \mathcal{W}N \implies \mathcal{W}(M[N]) \wedge (hgt\ M) - 1 \le hgt\ M[N]$$

Substitution

Substitution is defined in terms of instantiation:

$$M[x \leftarrow P] := (\mathsf{lam}_x M)[P]$$

- All the expected properties hold.
- Usual substitution lemma:

$$x \neq y \wedge x \notin \mathsf{FV}(\mathsf{P}) \wedge \mathcal{W}(\mathsf{M}, \mathsf{P}, \mathsf{N}) \implies$$
$$M[x \leftarrow N][y \leftarrow P] = M[y \leftarrow P][x \leftarrow N[y \leftarrow P]]$$

β reduction as usual

Using abstraction we have a natural definition of β reduction:

$$\frac{\mathcal{W}M \quad \mathcal{W}N}{\lceil \mathsf{lam}_x M, N \rceil_0 \overset{\beta}{\to} M[x \leftarrow N]} \quad (\beta)$$

$$\frac{M \overset{\beta}{\to} M' \quad \mathcal{W}N}{\lceil M, N \rceil_0 \overset{\beta}{\to} \lceil M', N \rceil_0} \qquad \frac{\mathcal{W}M \quad N \overset{\beta}{\to} N'}{\lceil M, N \rceil_0 \overset{\beta}{\to} \lceil M, N' \rceil_0}$$

$$\frac{M \overset{\beta}{\to} N}{\mathsf{lam}_x M \overset{\beta}{\to} \mathsf{lam}_x N} \quad (\xi)$$

- Any preterm that participates in this relation is well-formed.
- Correct β reduction w.r.t. the meaning of terms given above,
- Still contains rule ξ

Properties of usual β reduction

- As usual, rule ξ is invertible:

$$\mathsf{lam}_x M \stackrel{\beta}{\to} \mathsf{lam}_x N \implies M \stackrel{\beta}{\to} N$$

- β reduction does not lower height:

$$M \stackrel{\beta}{\to} N \implies hgt\, M \le hgt\, N$$

Generalized lifting

To eliminate rule ξ from our presentation of β reduction, we define *generalized lifting*.

$$(X_n y)^{i\Uparrow} := X_{n+1} y$$
$$(J_n j)^{i\Uparrow} := \begin{cases} J_{n+1} j & (j < i) \\ J_{n+1} (j+1) & (j \geq i) \end{cases}$$
$$(\lceil M, N \rceil_n)^{i\Uparrow} := \lceil (M)^{i\Uparrow}, (N)^{i\Uparrow} \rceil_{n+1}$$

- Preserves well formedness and raises height by one.
- Many useful properties of generalized lifting are used, e.g.
  - Injectivity: $\mathcal{W}(M, N) \wedge (M)^{i\Uparrow} = (N)^{i\Uparrow} \implies M = N$.

We iterate generalized lifting:

$$(M)^{i\Uparrow 0} := M$$
$$(M)^{i\Uparrow m+1} := ((M)^{i\Uparrow m})^{i\Uparrow}$$

Generalized instantiation

Generalized instantiation, $M[N]^i$, leaves terms M of height 0 unchanged, and updates abstractions:

$$X_{n+1}\, y[M]^i := X_n\, y$$
$$J_{n+1}\, i[M]^i := (M)^{i \Uparrow n-i}$$
$$J_{n+1}\, j[M]^i := \begin{cases} J_n\, j & (j < i) \\ J_n\, (j-1) & (j > i) \end{cases}$$
$$\lceil P,\, Q \rceil_{n+1}[M]^i := \lceil P[M]^i,\, Q[M]^i \rceil_n$$

- $A[P]^0 = A[P]$
- $n < hgt\, A \wedge n \le hgt\, P \implies n \le hgt\, (A[P]^n)$
- $n < hgt\, A \wedge n \le hgt\, P \wedge \mathcal{W}A \wedge \mathcal{W}P \implies \mathcal{W}(A[P]^n)$

β without rule ξ

Claim the relation $\bullet > \bullet$ defined without a ξ rule:

$$\frac{\mathcal{W}A \quad n < hgt\,A \quad \mathcal{W}N \quad n \leq hgt\,N}{\lceil A, N \rceil_n > A[N]^n} \quad (\beta)$$

$$\frac{M > M' \quad n \leq hgt\,M \quad \mathcal{W}N \quad n \leq hgt\,N}{\lceil M, N \rceil_n > \lceil M', N \rceil_n}$$

$$\frac{N > N' \quad n \leq hgt\,N \quad \mathcal{W}M \quad n \leq hgt\,M}{\lceil M, N \rceil_n > \lceil M, N' \rceil_n}$$

is equivalent to the relation $\bullet \xrightarrow{\beta} \bullet$ given above (and thus to the usual notion of β reduction).

Proof that $M > N \Longrightarrow M \xrightarrow{\beta} N$: by induction on the relation $M > N$.

Both congruence rule cases use invertibility of rule ξ for relation $\xrightarrow{\beta}$.

The converse direction is straightforward. $\qquad\qquad\square$

Tait–Martin-Löf parallel reduction: Usual presentation

Parallel reduction (non-deterministic):

$$\frac{}{X_0\, x \overset{p}{\to} X_0\, x} \qquad \frac{M \overset{p}{\to} M' \quad N \overset{p}{\to} N'}{\lceil \mathsf{lam}_x M,\, N \rceil_0 \overset{p}{\to} M'[x \leftarrow N']} \ (\beta)$$

$$\frac{M \overset{p}{\to} N}{\mathsf{lam}_x M \overset{p}{\to} \mathsf{lam}_x N} \ (\xi) \qquad \frac{M \overset{p}{\to} M' \quad N \overset{p}{\to} N'}{\lceil M,\, N \rceil_0 \overset{p}{\to} \lceil M',\, N' \rceil_0}$$

- Correct w.r.t. usual presentation.
- Overlap between rule (β) and application congruence.

Complete development (deterministic, à la Takahashi):

- Remove overlap, forcing every β step to be taken:

$$\frac{M \overset{cd}{\to} M' \quad N \overset{cd}{\to} N' \quad \textcolor{red}{M \text{ not an abstraction}}}{\lceil M,\, N \rceil_0 \overset{cd}{\to} \lceil M',\, N' \rceil_0}$$

Parallel reduction without rule ξ

Parallel reduction:

$$\frac{}{\mathsf{X}_n \, y \gg \mathsf{X}_n \, y} \qquad \frac{j < n}{\mathsf{J}_n \, j \gg \mathsf{J}_n \, j}$$

$$\frac{n \leq hgt \, M \quad M \gg M' \quad n \leq hgt \, N \quad N \gg N'}{\lceil M, N \rceil_n \gg \lceil M', N' \rceil_n}$$

$$\frac{n < hgt \, A \quad A \gg B \quad n \leq hgt \, M \quad M \gg N}{\lceil A, M \rceil_n \gg B[N]^n}$$

Complete development (remove overlap):

$$\frac{n = hgt \, M \quad M \ggg M' \quad n \leq hgt \, N \quad N \ggg N'}{\lceil M, N \rceil_n \ggg \lceil M', N' \rceil_n}$$

Church–Rosser theorem

- With parallel reduction and complete development, we can carry out Takahashi's proof of Church–Rosser.
- Although there is no rule ξ, this proof is no easier than usual.

η?

Consider a standard representation of pure η reduction:

$$\frac{\mathcal{W}M \quad x \notin \mathsf{FV}(M)}{\mathsf{lam}_x \lceil M, (X_0\, x) \rceil_0 \stackrel{\eta}{\to} M} \quad (\eta)$$

$$\frac{M \stackrel{\eta}{\to} M' \quad \mathcal{W}N}{\lceil M, N \rceil_0 \stackrel{\eta}{\to} \lceil M', N \rceil_0} \quad \frac{\mathcal{W}M \quad N \stackrel{\eta}{\to} N'}{\lceil M, N \rceil_0 \stackrel{\eta}{\to} \lceil M, N' \rceil_0} \quad \frac{M \stackrel{\eta}{\to} N}{\mathsf{lam}_x M \stackrel{\eta}{\to} \mathsf{lam}_x N}$$

Rule ξ is not invertible for this relation:

- $\mathsf{lam}_x \lceil \mathsf{lam}_x \lceil \overline{x}, \overline{x} \rceil_0, \overline{x} \rceil_0 \stackrel{\eta}{\to} \mathsf{lam}_x \lceil \overline{x}, \overline{x} \rceil_0$,
  but not $\lceil \mathsf{lam}_x \lceil \overline{x}, \overline{x} \rceil_0, \overline{x} \rceil_0 \stackrel{\eta}{\to} \lceil \overline{x}, \overline{x} \rceil_0$
- We might conjecture a ξ-free system for η, but our proof of correctness (using invertibility of ξ) will fail.
- $\stackrel{\eta}{\to}$ can reduce height, which the previous relations cannot do.