SIZED TYPES

TOWARDS TYPE-BASED TERMINATION THAT DOESN'T GET IN THE WAY OF EQUALITY

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CHALMERS

WHY SIZED TYPES?

SIZES CAN GET IN THE WAY OF EQUALITY

OUR PROPOSAL

WHY SIZED TYPES? STRUCTURAL ORDER LACKS COMPOSITIONALITY

```
sub : ℕ → ℕ → ℕ
sub zero     m        = zero
sub (suc n) zero     = suc n
sub (suc n) (suc m) = sub n m
```

WHY SIZED TYPES? STRUCTURAL ORDER LACKS COMPOSITIONALITY

```
div : ℕ → ℕ → ℕ
div zero m = zero
div (suc n) m = suc (div (sub n m) m)
```

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div : ℕ → ℕ → ℕ
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Termination checking failed!

WHY SIZED TYPES? THE NATURAL NUMBERS EXAMPLE

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data Nat : Size → Set where
  zero : ∀ i → Nat (↑i)
  suc  : ∀ j → (n : Nat j) → Nat (↑j)
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data Nat : Size → Set where
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```

```
sub : ∀ i → Nat i → Nat ∞ → Nat i
sub .(↑i) (zero i)  m          = (zero i)
sub .(↑i) (suc i n) (zero ∞)   = suc i n
sub .(↑i) (suc i n) (suc ∞ m) = sub i n m
```

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SIZES IN THE WAY OF EQUALITY EXAMPLE

```
gscale : (∀ i → Nat i → Nat i) → (Nat ∞ → Nat ∞) →
         ∀ i → Nat i → Nat ∞
gscale f g .(↑i) (zero i)   = zero ∞
gscale f g .(↑i) (suc i n) = g (suc ∞ (gscale f g i (f i n)))
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Generalization of `div`

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```
div ≅ gscale sub (λ x → x)
```

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```
scale1 = gscale (λ _ x → x) (λ x → x)
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```

```
scale1-id : ∀ i → (n : Nat i) → scale1 i n ≡ n
scale1-id .(↑i) (zero i)   = ?1
scale1-id .(↑i) (suc i n) = ?2
```

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```
scale1-id : ∀ i → (n : Nat i) → scale1 i n ≡ n
scale1-id .(↑i) (zero i)   = ?₁
scale1-id .(↑i) (suc i n) = ?₂
```

```
goal₁ zero ∞ ≡ zero i
```

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⇒ overhead with the proofs

⇒ no arbitrary rank size quantification

OUR PROPOSAL CUMULATIVITY AND BUILT-IN IRRELEVANCE

We feature

Arbitrary rank (first-class) size quantification

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Natural numbers with sizes irrelevant for equality

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```
zero a ≡ zero b : Nat c   for a,b < c
```

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Natural numbers with sizes irrelevant for equality

$$\texttt{zero}\ a \equiv \texttt{zero}\ b : \texttt{Nat}\ c \quad \text{for } a, b < c$$

Cumulativity for Sized Types

$$\frac{a \leq b}{\texttt{Nat}\ a \leq \texttt{Nat}\ b}$$

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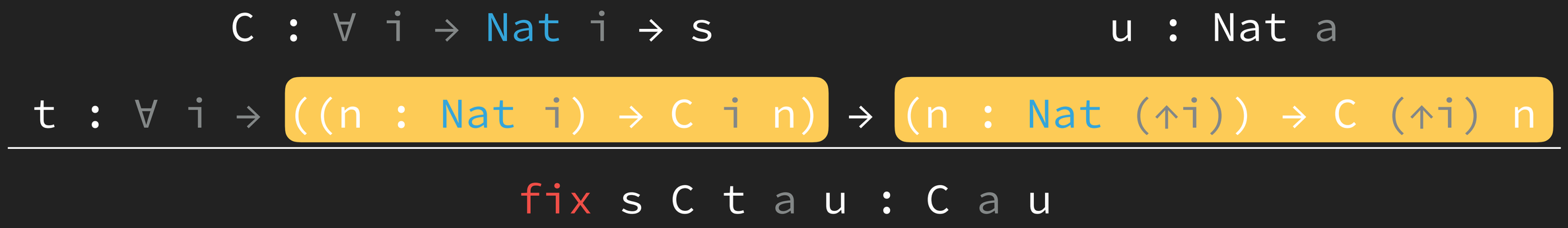
Cumulativity for Sized Types

$$\frac{a \leq b}{\texttt{Nat}\ a \leq \texttt{Nat}\ b} \qquad \frac{U = U' \qquad x{:}U \vdash T \leq T'}{(x{:}U) \to T \leq (x{:}U') \to T'}$$

OUR PROPOSAL TERMINATION

Ensured by fixed point operator



OUR PROPOSAL TERMINATION

Ensured by fixed point operator

$$\frac{C : \forall\ i \to \mathsf{Nat}\ i \to s \qquad u : \mathsf{Nat}\ a \qquad t : \forall\ i \to ((n : \mathsf{Nat}\ i) \to C\ i\ n) \to (n : \mathsf{Nat}\ (\uparrow i)) \to C\ (\uparrow i)\ n}{\mathsf{fix}\ s\ C\ t\ a\ u : C\ a\ u}$$

```
fix s C t (↑a) (zero b)  ↦ t a (λ (n : Nat a) → fix s C t a n) (zero b)
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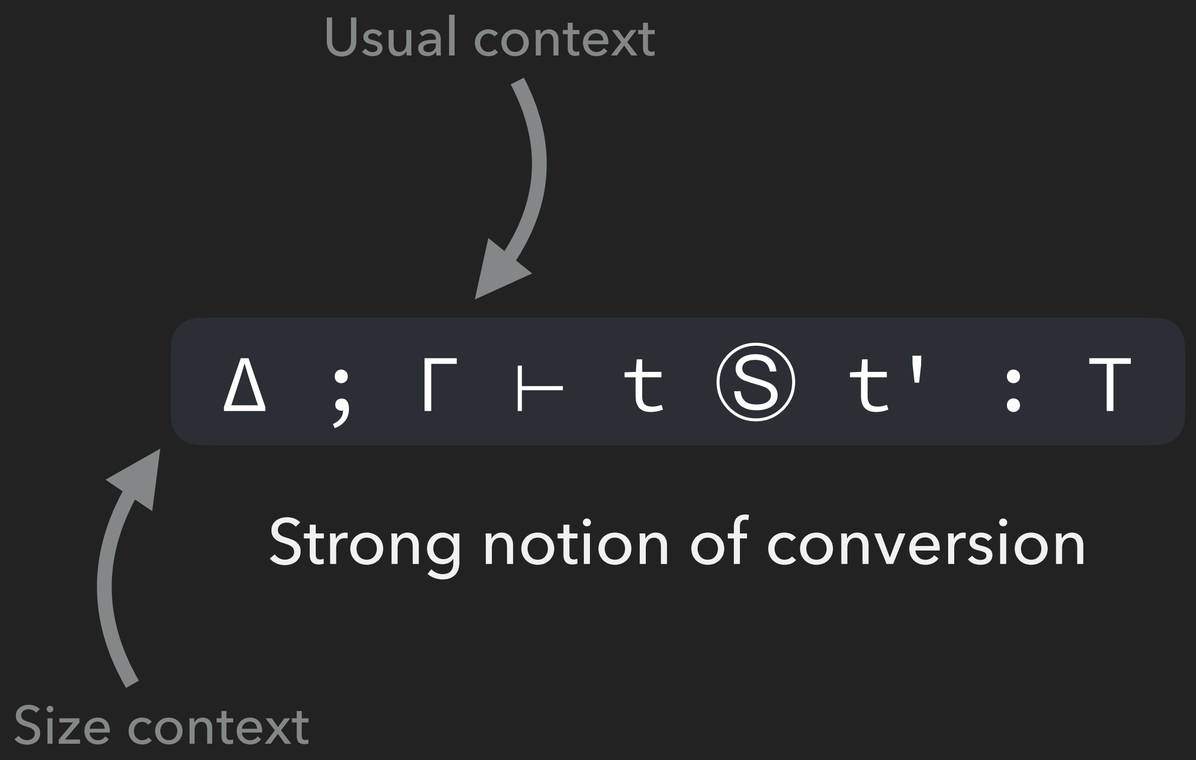
```
fix s C t (↑a) (zero b)  ↦ t a (λ (n : Nat a) → fix s C t a n) (zero b)
fix s C t (↑a) (suc b u) ↦ t a (λ (n : Nat a) → fix s C t a n) (suc b u)
```

OUR PROPOSAL MODEL / KRIPKE LOGICAL RELATION



OUR PROPOSAL KRIPKE LOGICAL RELATION (OUTLINE)

$$\frac{\forall\ \Delta' \leq \Delta,\ \Delta' \vdash a \Rightarrow \Delta' \ ;\ \Gamma \vdash f\ a \mathbin{\widehat{\textcircled{S}}} f'\ a : T[a/i]}{\Delta \ ;\ \Gamma \vdash f \mathbin{\textcircled{S}} f' : \forall\ i \to T}$$

OUR PROPOSAL KRIPKE LOGICAL RELATION (OUTLINE)

$$\frac{\frac{T \searrow A \quad \Delta;\Gamma \vdash T = A \qquad t \searrow a \quad t' \searrow a'}{\Delta;\Gamma \vdash t = a : A \quad \Delta;\Gamma \vdash t' = a' : A}}{\Delta;\Gamma \vdash a \mathbin{Ⓢ} a' : A}}{\Delta;\Gamma \vdash t :=: t' : T}$$

$$\Delta ; \Gamma \vdash t \mathbin{\widehat{Ⓢ}} t' : T$$

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$$\Delta ; \Gamma \vdash t \mathbin{\widehat{Ⓢ}} t' : T$$

⇒ normalization and subject reduction

OUR PROPOSAL ALGORITHMIC EQUALITY (OUTLINE)

Type-directed Equality

$$\frac{\Delta;\Gamma,\ x{:}U \vdash t\ x \Leftrightarrow t'\ x : T}{\Delta;\Gamma \vdash t \Leftrightarrow t' : (x{:}U) \to T}$$

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Structural Equality for neutrals

$$\frac{(x{:}U) \in \Gamma}{\Delta;\Gamma \vdash x \Leftrightarrow x : U}$$

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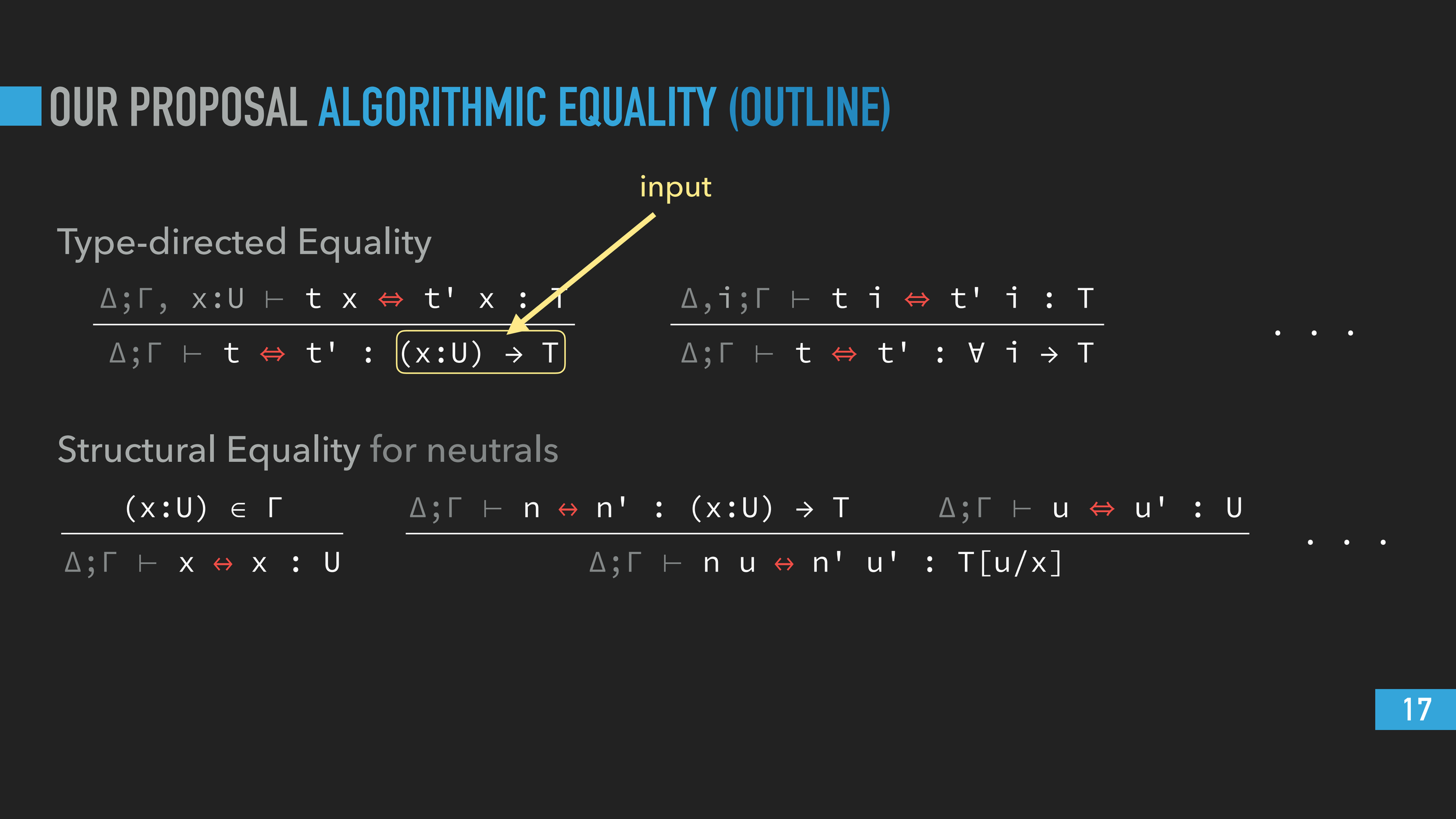

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input

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output

CONCLUSION

Decidable type system with

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Sized dependent types

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Type-based termination checking

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Only `Nat` for now

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Future work

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HoTT