Hybrid realizability for intuitionistic and classical choice

Valentin Blot

University of Bath

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Introduction

- Intuitionistic choice:
  - Easy interpretation in Kleene's/Kreisel's realizability
  - Provable in Martin-Löf's type theory
  - Natural interpretation of strong existentials

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  - Complex computational interpretation (bar recursion)
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  - Straightforward interpretation but intuitionistic logic only
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- 3 choices:
  - Straightforward interpretation but intuitionistic logic only
  - Full classical logic but complex interpretation
  - Both weak and strong existentials in one framework

Intuitionistic choice

$$\forall x \exists y\, A(x, y) \implies \exists f \forall x\, A(x, f\, x)$$

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$$\forall x \exists y\, A(x, y) \Longrightarrow \exists f \forall x\, A(x, f\, x)$$

- $M \Vdash \forall x \exists y\, A(x, y)$

$$\begin{array}{rrcl} \text{for any } x, & M\, x & \Vdash & \exists y\, A(x, y) \\ \text{for any } x, & \pi_2\,(M\, x) & \Vdash & A(x, \pi_1\,(M\, x)) \end{array}$$

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✓ Easy!

Classical choice

- Proves the existence of non-computable functions:

$$A \vee \neg A \vdash \forall x \exists y \, (y = 0 \Leftrightarrow A(x))$$

$$A \vee \neg A + AC \vdash \exists f \forall x \, (f\ x = 0 \Leftrightarrow A(x))$$

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✗ Much harder!

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No!

$$\pi_1\,M = 1 \qquad \text{and} \qquad \pi_2\,M = \texttt{refl}$$

$$\texttt{refl} \quad \not\Vdash \quad (1 = 0)$$

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Realizing classical choice

$$\texttt{barrec} \Vdash \forall x \neg\neg\exists y\, A\,(x, y) \Longrightarrow \neg\neg\exists f \forall x\, A\,(x, f\, x)$$

Realizability interpretation

- Classical logic:
  - $|A| = \{\text{realizers of } A\}$
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Strong existentials + classical logic = ☠

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- Multiple conclusions, a principal one:

$$A_1, \ldots, A_n \vdash B \mid C_1^-, \ldots, C_p^-$$

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- Classical reasoning restricted to negative formulas:

$$A^-, B^- \quad ::= \quad t \neq u \mid \bot \mid A \Rightarrow B^- \mid A^- \wedge B^- \mid \forall x\, A^-$$

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- No classical reasoning on strong existentials
- $\neg\neg\exists x\, A$ negative $\rightsquigarrow$ classical reasoning on weak existentials

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  take $B\,(x) \equiv \exists y\, A\,(x, y)$ and use intuitionistic choice to get:

  $$\varphi \in |\forall x \neg\neg\exists y\, A\,(x, y) \Longrightarrow \neg\neg\exists f \forall x\, A\,(x, f\,x)|$$

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  - $(\exists x\, A)^c = \neg\neg\exists x\, A^c$
  - $(\clubsuit\, A)^c = \clubsuit\, A^c$ for other connectives
- We can also mix strong and weak existentials
  - A classical proof can contain an intuitionistic subproof
  - The corresponding subroutine benefits from the efficiency of strong existentials (no backtrack)

Extraction

- From strong existentials:

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- From weak existentials:

If $\varphi \in |\neg\neg\exists x\,(t = u)|$, then $\mu\kappa.\varphi\,(\lambda y.\pi_2\, y\,([\kappa]\,\pi_1\, y)) \succ^* n$ with:

$$t\,[n/x] = u\,[n/x]$$

Conclusion & future works

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What we'll have:

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- What are these polarities?

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