Higher Categories in Homotopy Type Theory

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As homotopy type theory is viewed as a possible foundation of mathematics, it is natural to ask whether it can be used to develop category theory. This question has already been considered frequently. As we know, ordinary category theory can indeed be done in a nice way [1]. Given that types carry the structure of ∞-groupoids, it is not surprising that ordinary categories are often not sufficient. In particular, the universe itself is *not* a category in the sense of [1]. Therefore, what we want is a theory of $(\infty, 1)$-categories (simply referred to as ∞-*categories*). This is partially explored by Cranch [4], however only *concrete* categories (whose higher structure is reflected in the universe) are covered.

We expect that a general theory of ∞-categories has many applications in homotopy type theory. As equalities stated internally carry structure, different equalities are a priori not necessarily *coherent*. More often than not, this coherence will however be necessary for further constructions. While it is sometimes possible, it often does not seem feasible to handle a potentially huge number of coherence conditions manually. An example and main motivation for our development is the the project that develops a syntactical theory of higher inductive types, pursued by Dijkstra, Nordvall Forsberg, and two of the current authors [2] and is based on the semantics for higher inductive types described by Lumsdaine and Shulman [7]. The authors work with generalised containers, container algebras, and algebra morphisms, but the presentation is rendered extremely cumbersome by the fact that all the categorical laws only hold up to homotopy. For any given representation of a higher inductive type, stated as a list of constructors, the number of coherences that need to be considered is finite. In principle, this should allow the construction to go through; however, in practice, the sheer amount of these coherences cannot be handled manually in all but the most trivial cases. With a proper framework for ∞-categories, we expect that we get a clean way of resolving this problem.

A standard model of ∞-categories in set theory are Kan simplicial sets. This is essentially the notion of ∞-categories that we want to use, replacing sets by types. Note that we do not crucially insist on having a *type* of ∞-categories. Unless we extend homotopy type theory by some "infinitary" construction, this would in fact be an unreasonable expectation akin to the famous open problem of defining semisimplicial types. This is an important difference compared to the development of categories in the sense of [1]. What we want to settle for instead is a more "external" notion of ∞-category which will however have a *type* of cells on any given level. A framework in which this can be formalised would be Voevodsky's homotopy type system [9] or our 2-level theory [3]. Not having a type of ∞-categories does not seem to be a problem for typical applications. For example in the described project of developing a syntactical theory of higher inductive types, it will be sufficient to extract a *finite* number of coherence conditions from the ∞-categorical considerations; and such a finite collection will form a type.

As a further simplification, we choose to drop the requirement of degeneracies (or identities). This seems to be fine for the application of handling coherences (in further work, we will also investigate the possibility to add degeneracies in the way presented by Harpaz [5]). The advantage is that we can then consider type-valued contravariant diagrams over the *direct* category Δ_+, the category of finite ordinals and strictly increasing functions. In particular, we can consider Reedy fibrant diagrams, which can be described inductively (see [8]). This also ensures that it is reasonable to ask for *strict* semisimplicial laws.

In detail, let us write Type for the (strict, non-internally formulated) category of types and functions. Let $D : \Delta_+^{\mathrm{op}} \to \mathsf{Type}$ be a Reedy fibrant functor. Define $\mathsf{Sp}^D : \Delta_+^{\mathrm{op}} \to \mathsf{Type}$ to be the *nerve* or *spine* functor of D, with $\mathsf{Sp}_n^D :\equiv D_1 \times_{D_0} \ldots \times_{D_0} D_1$. Borrowing the usual terminology, we can say that D is a *semi-Segal type* if the canonical fibrations $D_n \twoheadrightarrow \mathsf{Sp}_n^D$ are all equivalences. We can then show that D is a semi-quasicategory if and only if it is a semi-Segal type (essentially by the same argument as exhibited in [6]).

Let us outline some examples of ∞-semicategories. First, consider any type A, and the diagram $\Delta_+^{\mathrm{op}} \to \mathsf{Type}$ which is constantly A. In [6], a fibrant replacement of this diagram is constructed explicitly. This yields indeed an ∞-semicategory, called the *equality semisimplicial type* $\mathcal{E}A$ in [6]. Not surprisingly, it fulfils the stronger property of being an ∞-*semigroupoid*.

Second, consider the family $D : \mathbb{N} \to \mathsf{Type}$, with $D_0 :\equiv \mathcal{U}$ (i.e. elements are small types), $D_1 :\equiv \Sigma\,(X_0, X_1 : \mathcal{U})\,.\, X_0 \to X_1$ (pairs of types and a function between them), $D_2 :\equiv \Sigma\,(X_0, X_1, X_2 : \mathcal{U})\,.\,(X_0 \to X_1) \times (X_1 \to X_2)$ (a chain $X_0 \to X_1 \to X_2$ of types), and so on; in general, D_n are chains of length n. D can be completed to a functor $\Delta_+^{\mathrm{op}} \to \mathsf{Type}$. Exactly as before, we can then take a Reedy fibrant replacement of this functor which, by construction, is a semi-Segal type. It corresponds to the ∞-semicategory of types and we call it TYPE.

In the sketched situation, we are lucky: completing the family $D : \mathbb{N} \to \mathsf{Type}$ to an actual functor from Δ_+^{op} is straightfoward as associativity of function composition holds strictly. Unfortunately, not all ∞-semicategories of interest can be dealt with in this fashion, since a similar construction would yield a family of types D that cannot be regarded as a strict functor in any obvious way. This already happens in the case of *pointed types*, where the analogous approach would be to start with chains of pointed types and pointed maps. However, we can instead consider chains of types and functions (as before) and a point only in the *first* type of the chain. This is an equivalent representation which carries a strict structure, giving rise to the ∞-semicategory PTYPE.

From the basic ingredients TYPE and PTYPE, and the fibration $\mathsf{PTYPE} \twoheadrightarrow \mathsf{TYPE}$, we can construct more sophisticated examples. The simplest interesting example is taking the local exponential of PTYPE with itself in context TYPE, which leads essentially to algebras over the identify functor. This way, we hope to achieve a reasonably theory of ∞-semicategories in homotopy type theory which can then also be used to treat coherences in a principled way.

References

[1] Benedikt Ahrens, Krzysztof Kapulkin, and Michael Shulman. Univalent categories and the Rezk completion. *Mathematical Structures in Computer Science (MSCS)*, pages 1–30, Jan 2015.

[2] Thorsten Altenkirch, Paolo Capriotti, Gabe Dijkstra, and Fredrik Nordvall Forsberg. Towards a theory of higher inductive types. Presentation at TYPES'15, Tallinn, Estonia, 20 May 2015.

[3] Thorsten Altenkirch, Paolo Capriotti, and Nicolai Kraus. Extending homotopy type theory with strict equality. *ArXiv e-prints*, April 2016.

[4] James Cranch. Concrete categories in homotopy type theory. *ArXiv e-prints*, Nov 2013.

[5] Yonatan Harpaz. Quasi-unital ∞–categories. *Algebraic & Geometric Topology*, 15(4):2303–2381, 2015.

[6] Nicolai Kraus. The general universal property of the propositional truncation. In *20th International Conference on Types for Proofs and Programs (TYPES 2014)*, *LIPIcs* vol. 39, pages 111–145, 2015.

[7] Peter LeFanu Lumsdaine and Michael Shulman. Semantics of higher inductive types, 2013. Unpublished note, `ncatlab.org/homotopytypetheory/files/hit-semantics.pdf`.

[8] Michael Shulman. Univalence for inverse diagrams and homotopy canonicity. *Mathematical Structures in Computer Science*, pages 1–75, Jan 2015.

[9] Vladimir Voevodsky. A simple type system with two identity types, 2013. Unpublished note.