Hybrid realizability for intuitionistic and classical choice*

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Realizability appeared in [Kle45] as a formal account of the Brouwer-Heyting-Kolmogorov interpretation of logic, leading to the Curry-Howard isomorphism between intuitionistic proofs and purely functional programs. In [Gri90], Griffin used control operators to extend the isomorphism to classical logic. While the first realizability interpretations of classical logic relied on a negative translation followed by an intuitionistic realizability interpretation, Griffin's discovery can be exploited to give a direct realizability interpretation of classical logic. This allowed Krivine to interpret second-order Peano arithmetic and the axiom of dependent choice in an untyped λ-calculus extended with the `call/cc` operator [Kri09]. In the work presented here, we interpret first-order classical arithmetic and the axiom of countable choice in a model of the simply-typed $\lambda\mu$-calculus [Par92], an extension of λ-calculus with control features.

The axiom of choice is ubiquitous in mathematics and is often used without even noticing. Therefore, having a computational interpretation of this axiom is essential to the extraction of programs from a wide range of mathematical proofs. While in usual interpretations of intuitionistic logic the general axiom of choice is trivially realized, interpreting even its countable version in a classical setting requires the use of strong recursion schemes, like the bar recursion operator [Spe62]. The requirement for such a strong recursion principle can be explained by the much stronger provability strength of classical choice: since any formula can be reflected by a boolean in classical logic, the axiom of countable choice can build the characteristic function of any formula with integer parameters, even the undecidable ones.

Variants of bar recursion were used in [BBC98, BO05] to interpret the negative translation of the axiom of choice in an intuitionistic setting. In [BR13], it was shown that bar recursion can be used in a language with control operators to interpret directly the axiom of countable choice in a classical setting. In the present work we extend this approach, adding strong existentials to the realizability interpretation. Because classical proofs typically involve some backtracking, strong existentials are problematic in a classical setting [Her05]. The reason is that the witness of a strong existential may change when the exploration of the associated proof performs some backtrack. Conversely, weak existentials (which, in our setting, are double negations of the strong ones) work well with control operators, but are less efficient from the computational perspective because their interpretations indeed involve backtracking.

Rather than having to choose between an efficient system which is restricted to intuitionistic logic or a full classical system with a complex computational interpretation, we take the best from both worlds and work within classical logic with strong existential quantifications, using their weak counterparts when classical reasoning is needed. In a proof where the excluded middle is never used on some existential formula, this existential can be strong and benefit from an efficient interpretation (in particular, the general axiom of choice is trivially realized in that case). If in the same proof some classical reasoning is performed on another existential formula, then that existential must be weak, and while we can still use the axiom of countable choice on it, its computational interpretation is given by bar recursion and can involve a costly recursion. Restricting strong existentials to intuitionistic logic relies on the polarities of [Blo15] which forbids classical reasoning on strong existentials and ensures the correctness of our computational interpretation in $\lambda\mu$-calculus. Similarity between our polarities and those of other

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proof systems like Girard's LU [Gir93] and LC [Gir91], or Liang and Miller's PCL [LM13] are still to be investigated. Combination of strong existentials and classical logic was also investigated in [Her12], where strong existentials were weakened enough to make them compatible with classical logic while preserving countable and dependent choice. We believe that there are strong connections between the operational semantics of Herbelin's calculus and bar recursion.

In our setting, witnesses can be extracted from proofs of strong existentials, as well as from proofs of Π^0_2 formulas with a weak existential. This second case relies on a standard non-empty realizability interpretation of the false formula. Our system allows for extraction of more efficient programs than with the usual direct or indirect interpretations of classical logic, provided some care is taken to choose strong existentials whenever possible.

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