The quaternionic Hopf fibration in homotopy type theory

Ulrik Buchholtz and Egbert Rijke

Carnegie Mellon University, Pittsburgh, USA
{ulrikb,erijke}@andrew.cmu.edu

The book on homotopy type theory [5] presents a beginning of the field of synthetic homotopy theory, which carries out homotopy-theoretical constructions internally in type theory. This makes it possible to very directly reason about abstract homotopy theory using proof assistants for dependent type theories, and as such construes homotopy theory as a branch of logic.

One high point in the book is the construction of the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. Classically, this is obtained by the so-called Hopf construction from the multiplication of unit-length complex numbers (that is, the unit circle in the complex plane), but in homotopy type theory we can reason about the circle in another way, namely as a higher inductive type, generated by a point constructor base and a path constructor $\mathsf{loop} : \mathsf{base} = \mathsf{base}$.

The Hopf construction applies to a connected H-space A, that is, a path-connected type A together with a binary operation μ and an element e that is neutral for this operation up to homotopy: $\mu(x, e) = x = \mu(e, x)$ for $x : A$ (the equality symbol refers to the identity type). The result is a fibration $\mathsf{hopf} : \Sigma A \to \mathsf{Type}$, where ΣA denotes the suspension of A. The total space can then be identified with the join $A * A$ of A with itself (the join $A * B$ of two types A and B is defined as the pushout of the two projections from $A \times B$, taking advantage of the fact that a pushout in type theory denotes a homotopy pushout of the corresponding spaces). Note that the join of two spheres is again a sphere: $S^n * S^m \simeq S^{n+m+1}$.

The Hopf fibration is in fact but one of a family of four fundamental fibrations in homotopy theory built from the classical normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$ (the real numbers, the complex numbers, the quaternions and the octonions). From these we classically obtain H-space structures on the unit spheres, S^0, S^1, S^3, S^7, inside these algebras, and from these the corresponding Hopf fibrations:

$$
\begin{aligned}
S^0 &\hookrightarrow S^1 \to S^1 \\
S^1 &\hookrightarrow S^3 \to S^2 \\
S^3 &\hookrightarrow S^7 \to S^4 \\
S^7 &\hookrightarrow S^{15} \to S^8
\end{aligned}
$$

Coming back to HoTT, it has been an open problem to construct the quaternionic Hopf fibration corresponding to the multiplication of unit quaternions (as $\mathbb{H}$ is a 4-dimensional algebra, these form a three-sphere, S^3).

Here we present a solution to this problem based on a modification of the Cayley-Dickson construction of the normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$.

The Cayley-Dickson construction (as described for instance by Baez [1]) produces a new $*$-algebra $A' := A \oplus A$ from a given one A by the stipulations

$$(a, b)^* := (a^*, -b), \quad (a, b)(c, d) := (ac - db^*, a^*d + cb).$$

This procedure cannot be productively replicated in homotopy type theory for our purpose, as any vector space over the reals is contractible as a homotopy type.

We were able to find an analog in the setting of synthetic homotopy theory by concentrating on the *unit imaginaries* inside the classical $*$-algebras, and we give a construction that produces an H-space structure on $\Sigma A * \Sigma A$ for any type A with an involutive negation for which the suspension ΣA has an associative H-space structure that interacts nicely with the negation on A.

Applying the construction to the 0-sphere (and thus to the multiplication on its suspension, the circle), we obtain an H-space structure on the three-sphere, and thus the quaternionic Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$. A famous application of the quaternionic Hopf fibration is as one ingredient in Milnor's construction of exotic 7-spheres [3], that is, smooth manifold structures on the S^7 that are not diffeomorphic to the standard smooth structure. This application, however, is not yet within reach of synthetic homotopy theory.

Our results have been formalized in the Lean proof assistant [4]. Lean has built-in support for HoTT in that it provides two kinds of higher inductive types, namely type quotients and n-truncations, from which a whole host of other common HITs can be deduced, including pushouts, joins, spheres, etc.

The formalization makes heave use of the cubical methods developed in [2], which leverage indexed inductive types representing squares and cubes in types, and paths, squares and cubes lying over given paths, squares and cubes in type families.

References

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