FLABloM: Functional linear algebra with block matrices

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In [1] Bernardy & Jansson used a recursive block formulation of matrices to certify Variant's [4] parsing algorithm. Their matrix formulation was restricted to matrices of size $2^n \times 2^n$ and this work extends the matrix formulation to allow for all sizes of matrices and applies similar techniques to algorithms that can be described as transitive closures of semi-rings of matrices with inspiration from [2] and [3].

We define a hierarchy of ring structures as Agda records. A semi-near-ring for some type s needs an equivalence relation $\simeq_s$, a distinguished element 0_s and operations addition $+_s$ and multiplication $\cdot_s$. Our semi-near-ring requires that 0_s and $+_s$ form a commutative monoid (i.e. $+_s$ commutes and 0_s is the left and right identity of $+_s$), 0_s is the left and right zero of $\cdot_s$, $+_s$ is idempotent ($\forall\ x \to x\ +_s\ x\ \simeq_s\ x$) and $\cdot_s$ distributes over $+_s$.

For the semi-ring we extend the semi-near-ring with another distinguished element 1_s and proofs that $\cdot_s$ is associative and that 1_s is the left and right identity of $\cdot_s$.

Finally we extend the semi-ring with an operation *closure* that computes the transitive closure of an element of the semi-ring (c is the closure of w if $c\ \simeq_s\ 1_s\ +_s\ w\ \cdot_s\ c$ holds), we denote the closure of w with w^*.

We use two examples of semi-rings with transitive closure: (1) the Booleans with disjunction as addition, conjunction as multiplication and the closure being *true*; and (2) the natural numbers ($\mathbb{N}$) extended with an element ∞, we let $0_s = \infty$, $1_s = 0$, *min* plays the role of $+_s$, addition of natural numbers the role of $\cdot_s$ and the closure is 0.

Matrices To represent the dimensions of matrices we use a type of non-empty binary trees:

```
data Shape : Set where
  L : Shape
  B : (s1 s2 : Shape) → Shape
```

This representation follows the structure of the matrix representation more closely than natural numbers and we can easily compute the corresponding natural number:

```
toNat : Shape → ℕ; toNat L = 1; toNat (B l r) = toNat l + toNat r
```

while the other direction is slightly more complicated because we want a somewhat balanced tree and we have no representation for 0.

Matrices are parametrised by the type of elements they contain and indexed by a *Shape* for each dimension. We use a datatype M with four constructors: *One*, *Row*, *Col*, and Q. The first *One* lifts an element into a 1-by-1 matrix:

```
data M (a : Set) : (rows cols : Shape) → Set where
  One :                 a → M a L L
```

Row and column matrices are built from smaller matrices which are either 1-by-1 matrices or further row respectively column matrices

$$Row : \{c_1\ c_2 : Shape\} \to M\ a\ L\ c_1 \to M\ a\ L\ c_2 \to M\ a\ L\ (B\ c_1\ c_2)$$
$$Col : \{r_1\ r_2 : Shape\} \to M\ a\ r_1\ L \to M\ a\ r_2\ L \to M\ a\ (B\ r_1\ r_2)\ L$$

and matrices of other shapes are built from 2×2 smaller matrices

$$\begin{aligned} Q : \{r_1\ r_2\ c_1\ c_2 : Shape\} \to\ & M\ a\ r_1\ c_1 \to M\ a\ r_1\ c_2 \to \\ & M\ a\ r_2\ c_1 \to M\ a\ r_2\ c_2 \to \\ & M\ a\ (B\ r_1\ r_2)\ (B\ c_1\ c_2) \end{aligned}$$

This matrix representation allows for simple formulations of matrix addition, multiplication, and as we will see also the transitive closure of a matrix.

Transitive closure In [3] Lehmann presents a definition of the closure on square matrices, $A^* = 1 + A \cdot A^*$: Given

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

the transitive closure of A is defined inductively as

$$A^* = \begin{bmatrix} A^*_{11} + A^*_{11} \cdot A_{12} \cdot \Delta^* \cdot A_{21} \cdot A^*_{11} & A^*_{11} \cdot A_{12} \cdot \Delta^* \\ \Delta^* \cdot A_{21} \cdot A^*_{11} & \Delta^* \end{bmatrix}$$

where $\Delta = A_{22} + A_{21} \cdot A^*_{11} \cdot A_{12}$ and the base case is the 1-by-1 matrix where we use the transitive closure of the element of the matrix: $[s]^* = [s^*]$.

We have encoded this definition of closure in Agda and implemented a constructive correctness proof using structural induction and equational reasoning. The full development of around 2500 lines of literate Agda code (including this abstract) is available on GitHub (https://github.com/DSLsofMath/FLABloM).

Conclusions We have presented an algebraic structure useful for (block) matrix computations and implemented and proved correctness of transitive closure. Compared to [1] our implementation handles arbitrary matrix dimensions but is restricted to semi-rings. Future work would be to extend the proof to cover both arbitrary dimensions and the more general semi-near-ring structure which would allow parallel parsing as an application.

References

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