If-then-else and other constructive and classical connectives

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Abstract

We develop a general method for deriving natural deduction rules from the truth table for a connective. The method applies to both constructive and classical logic. This implies we can derive "constructively valid" rules for any (classical) connective. We show this constructive validity by giving a general Kripke semantics, that is shown to be sound and complete for the constructive rules. For the well-known connectives, like $\vee$, $\wedge$, $\to$, the constructive rules we derive are equivalent to the natural deduction rules we know from Gentzen and Prawitz. However, they have a different shape, because we want all our rules to have a standard "format", to make it easier to define the notions of cut and to study proof reductions. In style they are close to the "general elimination rules" by Von Plato [4]. The rules also shed some new light on the classical connectives: e.g. the classical rules we derive for $\to$ allow to prove Peirce's law. Our method also allows to derive rules for connectives that are usually not treated in natural deduction textbooks, like the "if-then-else", whose truth table is clear but whose constructive deduction rules are not. We prove that "if-then-else", in combination with $\bot$ and $\top$, is functionally complete (all other constructive connectives can be defined from it). We define the notion of cut, generally for any constructive connective and we describe the process of "cut-elimination". Following the Curry-Howard isomorphism, we can give terms to deductions and we study cut-elimination as term reduction. We prove that reduction is strongly normalizing for constructive if-then-else logic.

Overview of the talk

Definition Suppose we have an n-ary connective c with a truth table t_c (with 2^n rows). We write $\phi = c(p_1, \ldots, p_n)$, where $p_1, \ldots, p_n$ are proposition letters and we write $\Phi = c(A_1, \ldots, A_n)$, where $A_1, \ldots, A_n$ are arbitrary propositions. Each row of t_c gives rise to an elimination rule or an introduction rule for c in the following way.

$$\begin{array}{ccc|c} p_1 & \ldots & p_n & \phi \\ \hline a_1 & \ldots & a_n & 0 \end{array} \quad \mapsto \quad \frac{\vdash \Phi \quad \ldots \vdash A_j \text{ (if } a_j = 1) \ldots \quad \ldots A_i \vdash D \text{ (if } a_i = 0) \ldots}{\vdash D} \text{el}$$

$$\begin{array}{ccc|c} p_1 & \ldots & p_n & \phi \\ \hline b_1 & \ldots & b_n & 1 \end{array} \quad \mapsto \quad \frac{\ldots \vdash A_j \text{ (if } b_j = 1) \ldots \quad \ldots A_i \vdash \Phi \text{ (if } b_i = 0) \ldots}{\vdash \Phi} \text{in}^i$$

$$\begin{array}{ccc|c} p_1 & \ldots & p_n & \phi \\ \hline c_1 & \ldots & c_n & 1 \end{array} \quad \mapsto \quad \frac{\Phi \vdash D \quad \ldots \vdash A_j \text{ (if } c_j = 1) \ldots \quad \ldots A_i \vdash D \text{ (if } c_i = 0) \ldots}{\vdash D} \text{in}^c$$

If $a_j = 1$ in t_c, then A_j occurs as a Lemma in the rule; if $a_i = 0$ in t_c, then A_i occurs as a Casus. The rules are given in abbreviated form and it should be understood that all judgments can be used with an extended hypotheses set Γ.

Example From the truth table we derive the following intuitionistic rules for $\wedge$, 3 elimination rules and one introduction rule:

$$\frac{\vdash A \wedge B \quad A \vdash D \quad B \vdash D}{\vdash D} \wedge\text{-el}_a \qquad \frac{\vdash A \wedge B \quad A \vdash D \quad \vdash B}{\vdash D} \wedge\text{-el}_b$$

$$\frac{\vdash A \wedge B \quad \vdash A \quad B \vdash D}{\vdash D} \wedge\text{-el}_c \qquad \frac{\vdash A \quad \vdash B}{\vdash A \wedge B} \wedge\text{-in}$$

These rules are all intuitionistically correct, as one can observe by inspection. We will show that these are equivalent to the well-known intuitionistic rules. We will also show how these rules can be optimized and be reduced to 2 elimination rules and 1 introduction rule.

From the truth table we also derive the following rules for $\neg$, 1 elimination rule and 1 introduction rule, a classical and an intuitionistic one.

$$\frac{\vdash \neg A \quad \vdash A}{\vdash D}\neg\text{-el} \qquad \frac{A \vdash \neg A}{\vdash \neg A}\neg\text{-in}^i \qquad \frac{\neg A \vdash D \quad A \vdash D}{\vdash D}\neg\text{-in}^c$$

As an example of the classical derivation rules we can show that $\neg\neg A \vdash A$ is derivable.

Contribution of the paper and related work

Natural deduction has been studied extensively, since the original work by Gentzen, both for classical and intuitionistic logic. Overviews can be found in [3] and [1]. Also the generalization of natural deduction to include other connectives or allow different derivation rules has been studied by various researchers. Notably, there is the work of Schroeder-Heister [2] and Von Plato [4] is related to ours. Schroeder-Heister studies general formats of natural deduction where also rules may be discharged (as opposed to the normal situation where only formulas may be discharged). He also studies a general rule format for intuitionistic logic and shows that the connectives $\wedge, \vee, \rightarrow, \bot$ are complete for it. Von Plato discusses "generalized elimination rules", which also appear naturally as a consequence of our approach of deriving the rules from the truth table.

However, we focus not so much on the rules but on the fact that we can define different and new connectives constructively. In our work, we derive the rules directly from the truth table and we give a complete Kripke semantics for the constructive connectives. This also allows us to prove some meta properties about the rules. For example, we give a generalization of the *disjunction property* in intuitionistic logic. We define and study cuts precisely, for the intuitionistic case. We look more in detail into the logic with just if-then-else and we prove that cut-elimination is strongly normalizing by studying the reduction of proof terms. We also show that if-then-else with $\bot$ and $\top$ is functionally complete for intuitionistic logic.

References

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