# Towards probabilistic reasoning about lambda terms with intersection types

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In the last 30 years several formal tools have been developed for reasoning about uncertain knowledge. One of these approaches concerns formalization in terms of probabilistic logics. Although the idea of probabilistic logic can be traced back to Leibnitz, Lambert and Boole, the modern development was started by Nils Nilsson, who tried to provide a logical framework for uncertain reasoning [7]. After Nilsson, a number of researchers proposed formal systems for probabilistic reasoning, for example [4], [5], [8].

Intersection types ([3]) were introduced in the lambda calculus as an extension of the simple types in order to overcome the limitations of the simple (functional) types. Indeed, lambda calculus with intersection types has two unique properties which do not hold in other type systems. First, it completely characterizes the termination of reduction, a.k.a strong normalization, in lambda calculus (e.g. [6]). Second, its type assignment is sound and complete with respect to the filter model, which was proven in the seminal paper by Barendregt et al. [1]. The latter result will be useful in this work.

We introduce in this paper a formal model $\mathsf{P}\Lambda^{\cap}$ for reasoning about probabilities of lambda terms with intersection types which is a combination of lambda calculus and probabilistic logic. We propose its syntax, Kripke-style semantics and an infinitary axiomatization. We first endow the language of typed lambda calculus with a probabilistic operator $P_{\geq s}$ and, besides the formulas of the form $M : \sigma$ and its Boolean combinations, we obtain formulas of the form

$$P_{\geq s} M : \sigma$$

to express that the probability that the lambda term $M$ is of type $\sigma$ is equal to or greater than $s$. More generally, formulas are of the form $P_{\geq s}\alpha$, where $\alpha$ is typed lambda statement $M : \sigma$ or its Boolean combination, so the following is a formula of our formal model as well:

$$[P_{=\frac{1}{3}}(x : \sigma \to \tau) \wedge P_{=\frac{2}{3}}(y : \sigma)] \Rightarrow [P_{=0}(xy : \tau) \vee P_{=\frac{1}{3}}(xy : \tau)].$$

We then propose a semantics of $\mathsf{P}\Lambda^{\cap}$ based on a set of possible worlds, where each possible world is a lambda model. The set of possible worlds is equipped with a probability measure $\mu$. The set $[\alpha]$ is the set of possible worlds that satisfy the formula $\alpha$. Then the probability of $\alpha$ is obtained as $\mu([\alpha])$. Finally, we give an infinitary axiomatization of $\mathsf{P}\Lambda^{\cap}$, which is a combination of deduction rules for lambda calculus with intersection types, axioms for the classical propositional logic, as well as the axioms for probabilistic logics, and prove the deduction theorem.

The main results we want to prove are the soundness and strong completeness of $\mathsf{P}\Lambda^{\cap}$ with respect to the proposed model, where strong completeness means that every consistent set of formulas is satisfiable. The construction of the canonical model is crucial for the proof and relays on two key facts. The first one is that the lambda calculus with intersection types is complete with respect to the filter lambda model, [1], and the second one is the property that every consistent set can be extended to the maximal consistent set.

In the last decade, several probabilistic extensions of the $\lambda$-calculus have been introduced and investigated. They are concerned with introducing non-determinism and probabilities into the syntax and operational semantics of the $\lambda$-calculus in order to formalize computation in the presence of uncertainty rather than with providing a framework that would enable probabilistic reasoning about typed terms and type assignments.

A slightly similar approach to ours, that provides a framework for probabilistic reasoning about typed terms, was treated by Cooper et al. in [2], where the authors proposed a probabilistic type theory in order to formalize computation with statements of the form "a given type is assigned to a given situation with probability $p$". However, the developed theory was used for analyzing semantic learning of natural languages in the domain of computational linguistics, and no soundness or completeness issues were discussed.

# References

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