On the Set Theory of Fitch-Prawitz* ABSTRACT

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The discovery of Russell's and Curry's Paradoxes inhibited mainstream research from exploring self-referential and inclusive Foundational Theories. The very concept of *type* was introduced precisely to disallow self-application, thus preventing itself from providing a *theory of everything*. In the last century, however, there have been isolated attempts to put forward some inclusive set theories. In 1952 Frederic B. Fitch [1] introduced a consistent Set-Theory, which compensates the effects of un-constrained abstraction by restricting the class of proofs. He introduced two possible restrictions which are rather idosyncratic and too restrictive[1]. It was not until Prawitz [7] gave a *natural deduction* presentation of Fitch's Theory, that a principled restriction on proofs was introduced, namely that the proof *be normal.*

Apart from this restriction on proofs, Fitch-Prawitz Set Theory, FP, is a standard first order theory with classical negation. Sets, *i.e.* abstractions, are introduced and eliminated in the natural way, and equality is expressed by *Leibniz equality.* The crucial rules are:

$$\lambda\text{I})\quad \frac{A[t/x]}{t \in \lambda x.A} \qquad\qquad \lambda\text{E})\quad \frac{t \in \lambda x.A}{A[t/x]} \qquad\qquad \bot)\quad \frac{\begin{matrix}(\neg A)\\ \vdots \\ \bot\end{matrix}}{A}$$

FP is provably consistent almost by definition. Since there is no introduction rule for $\bot$, it cannot be derived by a normal proof. Thus even if Russell's class, $R \equiv \lambda x.x \notin x$, can be defined, Russell's Paradox does not fire since the proof of contradiction is not *normalizable* and hence not valid. *Tertium non datur* holds but *Aristotle's non-contradiction pinciple* fails in that both $\vdash_{\mathsf{FP}} R \in R$ and $\vdash_{\mathsf{FP}} R \notin R$ are derivable. FP subsumes higher-order logic for all orders. A considerable part of the theory of real numbers can be developed in FP, although standard rules such as *modus ponens* or *extensionality* are not admissible.

The root reason for Russell's paradox is not *extensionality* or *tertium non datur*. Curry's paradox holds also in Minimal Logic with no use of extensionality. The two catastrophic ingredients are *unrestricted contraction* or *unrestricted proofs*. We get consistent theories when any of the two are guarded. Grishin [4] showed that extensionality implies contraction, but Girard in [3] showed that *Light Linear Logic*'s contraction yields a perfectly sensible, albeit weak Set-Theory. FP rules out contradictions by not allowing to introduce $\bot$ by brute force.

We discuss Fitch-Prawitz Set Theory *per se*, and give a *Fixed Point Theorem* whereby one can show that all provably total recursive functions are typable in FP.

Universal Set Theories which support extensionality have been occasionally introduced in the literature. One of the most inclusive is the Theory of *Hyperuniverses*, see [2]. Consistency is achieved by restricting the class of predicates which are allowed in the λ-*abstraction* rules, to *generalized positive formulæ*. These are defined as follows:

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[1] Fitch *simple restriction* does not validate $A \to (B \to ((A \wedge B) \to C) \to C))$, while *special restriction* does not permit $(P \leftrightarrow (P \to Q)) \to Q$ or $((A \to (A \to B) \to B) \to A) \to A$.

Definition 1 (Generalized Positive Formulæ)**.** *The* Generalized Positive Formulæ *are the smallest class of formulæ which*

- *include* $x \in y$, $x = y$;
- *are closed under the logical connectives* $\wedge, \vee$;
- *are closed under the quantifiers* $\forall x, \exists x, \forall x \in y, \exists x \in y$;
- *are closed under the formula* $\forall x.(\theta \to \phi)$, *where* ϕ *is a GPF and* $FV(\theta) = \{x\}$.

We provide an intriguing connection between FP and the Theory of Hyperuniverses. Namely we show that the strongly extensional collapse, *i.e. bisimilarity quotient*, of Fitch-Prawitz $\mathcal{P}$-coalgebra (V, f_{FP}), where V is the set of closed terms of FP and $f_{\mathsf{FP}}(t) = \{s \mid \ \vdash_{\mathsf{FP}} s \in t\}$, satisfies the astraction principle for *generalized positive formulæ*.

Finally, we show how to encode FP in the Logical Framework $\mathsf{LLF}_{\mathcal{P}}$ introduced in [6], using *monadic locks*. This work appears in [5]. $\mathsf{LLF}_{\mathcal{P}}$ is an extension of the Logical Framework LF which allows for delegating to an external tool the task of checking that a proof-term satisfies a given constraint. In the case of FP the constraint is that the proof term encodes a *normalizable proof.* This is, in fact, a slight generalization of the original system of Prawitz which allows for a *semi-decidable* notion of proof. The added value of using $\mathsf{LLF}_{\mathcal{P}}$ w.r.t. traditional LF is that the encoding can be shallower and hence more transparent. Taking `o` as the type of propositions and ι as the type of terms, the encoding requires to introduce two *judgements*, namely *valid* (`V:o->Type`) and *true* (`T:o->Type`). Only valid judgements can be assumed but only true judgements can be proved, whence a weaker form of $\to$-elimination can be expressed. For instance, if we consider the fragment of Fitch Prawitz Set Theory with $\to$ and the "membership" predicate ϵ as the constructors for propositions, we can introduce in the signature Σ_{FPST} the following constants:

```
lam: (ι -> o)-> ι        ϵ: ι -> ι -> o        →: o -> o -> o        δ: ΠA:o.(V(A) -> T(A))
λ_I: ΠA:ι ->o.Πx:ι.T(A x) -> T(ϵ x (lam A)) λ_E: ΠA:ι ->o.Πx:ι.T(ϵ x (lam A)) -> T(A x)
→ _I : ΠA,B:o.(V(A) -> T(B)) -> (T(A→B))
→ _E : ΠA,B:o.Πx:T(A).Πy:T(A→B) -> L^Fitch_{⟨x,y⟩,T(A)×T(A→B)}[T(B)]
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where `lam` is the "abstraction" operator for building "sets", δ is the coercion function, and $\langle \mathtt{x}, \mathtt{y} \rangle$ denotes the encoding of pairs. The predicate in the lock $\mathtt{Fitch}(\Gamma \vdash_{\Sigma_{\mathsf{FPST}}} \langle \mathtt{x}, \mathtt{y} \rangle\mathtt{:T(A)}\times\mathtt{T(A} \to \mathtt{B)})$ holds iff `x` and `y` have *skeletons* in $\Lambda_{\Sigma_{\mathsf{FPST}}}$ (*i.e.*, in the set of $\mathsf{LLF}_{\mathcal{P}}$ terms definable using constants from the signature Σ_{FPST}), all the holes of which have either type `o` or are guarded by a δ, and hence have type `V(A)`, and, moreover, the proof derived by combining the *skeletons* of `x` and `y` is normalizable in the natural sense. Clearly, this predicate is only semidecidable.

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