β reduction without rule ξ

Masahiko Sato and Randy Pollack

It is well known that, for β reduction of pure λ terms, the ξ rule is invertible:

$$\lambda x.s \xrightarrow{\beta} \lambda x.t \implies s \xrightarrow{\beta} t$$

With this observation we give a de Bruijn-like representation of pure λ terms, and rules for β reduction in this representation that need no rule ξ because rule ξ is admissible. This work has been formalized in Isabelle/HOL and proved adequate w.r.t. nominal Isabelle.

Fix a countable set of names, ranged over by x, y. Let i, j, m, n range over natural numbers. The raw syntax of preterms is

$$\mathsf{pt} ::= \ \mathsf{X}_n\,x \mid \mathsf{J}_n\,j \mid (M\ N)_n$$

Preterms are ranged over by M, N, P, Q, and indexed by their *height*, n (write $\mathit{hgt}\,M$). There is a notion of *well formedness* of preterms, $\mathcal{W}\,M$, defined inductively by

$$\frac{}{\mathcal{W}\,\mathsf{X}_n\,x} \qquad \frac{i < n}{\mathcal{W}\,\mathsf{J}_n\,i} \qquad \frac{\mathcal{W}\,M \quad \mathcal{W}\,N \quad n \le \mathit{hgt}\,M \quad n \le \mathit{hgt}\,N}{\mathcal{W}\,(M\ N)_n}$$

If $\mathcal{W}\,M$ we call M a *term*, and write $\mathcal{W}_n\,M$ to mean $\mathcal{W}\,M$ and $n \le \mathit{hgt}\,M$. The height of a term shows how many bindings it implicitly sits under.

We can define *abstraction* as a function on preterms:

$$\begin{aligned} \mathsf{lam}_x(\mathsf{X}_n\,y) &:= \text{ if } x = y \text{ then } \mathsf{J}_{n+1}\,0 \text{ else } \mathsf{X}_{n+1}\,y \\ \mathsf{lam}_x(\mathsf{J}_n\,j) &:= \mathsf{J}_{n+1}\,(j{+}1) \\ \mathsf{lam}_x((M\ N)_n) &:= (\mathsf{lam}_x(M)\ \mathsf{lam}_x(N))_{n+1} \end{aligned}$$

Abstraction preserves well formedness and raises height by one.

$$\mathcal{W}_n\,M \implies \mathcal{W}_{n+1}\,\mathsf{lam}_x(M)$$

Conversely, every term with height a successor is an abstraction. We use A, B as metavariables over abstractions.

The intended interpretation of preterms is given by the relation

$$x \sim \mathsf{X}_0\,x \qquad \frac{t_1 \sim M_1 \quad t_2 \sim M_2}{(t_1\,t_2) \sim (M_1\ M_2)_0} \qquad \frac{t \sim M}{\lambda x.t \sim \mathsf{lam}_x(M)}$$

which is an isomorphism between conventional λ terms (e.g. nominal terms) and terms of our formal lamguage.

To define instantiation we first introduce a lifting function

$$(\mathsf{X}_n\,y)^{\uparrow} := \mathsf{X}_{n+1}\,y \qquad (\mathsf{J}_n\,j)^{\uparrow} := \mathsf{J}_{n+1}\,(j{+}1) \qquad ((M\ N)_n)^{\uparrow} := ((M)^{\uparrow}\ (N)^{\uparrow})_{n+1}$$

which we iterate as: $(M)^{\uparrow 0} := M$ and $(M)^{\uparrow m+1} := ((M)^{\uparrow m})^{\uparrow}$.

Instantiation is a binary function, $M[N]$. If $\mathit{hgt}\,M = 0$ (M is under no binders), $M[N] = M$. Otherwise $M[N]$ fills any holes $\mathsf{J}_{n+1}\,0$ in M and adjusts the rest of the term:

$$\mathsf{X}_{n+1}\,y[N] := \mathsf{X}_n\,y \qquad \mathsf{J}_{n+1}\,0[N] := (N)^{\uparrow n} \qquad (M\ P)_{n+1}[N] := (M[N]\ P[N])_n$$
$$\mathsf{J}_{n+1}\,(j{+}1)[N] := \mathsf{J}_n\,j$$

Instantiation preserves well formedness and lowers height by one:

$$\mathcal{W}_{n+1}\, M \wedge \mathcal{W}\, N \implies \mathcal{W}_n\, M[N]$$

Using abstraction we have a natural definition of β reduction:

$$\frac{\mathcal{W}\, M \quad \mathcal{W}\, N}{(\mathsf{lam}_x(M)\ N)_0 \stackrel{\beta}{\to} (\mathsf{lam}_x(M))[N]}$$

$$\frac{M \stackrel{\beta}{\to} M' \quad \mathcal{W}\, N}{(M\ N)_0 \stackrel{\beta}{\to} (M'\ N)_0} \qquad \frac{\mathcal{W}\, M \quad N \stackrel{\beta}{\to} N'}{(M\ N)_0 \stackrel{\beta}{\to} (M\ N')_0} \qquad \frac{M \stackrel{\beta}{\to} N}{\mathsf{lam}_x(M) \stackrel{\beta}{\to} \mathsf{lam}_x(N)}$$

Any preterm that participates in this relation is well-formed. This relation is correct β reduction w.r.t. the meaning of preterms given above, but still contains an invertible ξ rule. To define an equivalent relation with no ξ rule we need to define *generalized lifting*, $(M)^{i\Uparrow}$:

$$(\mathsf{X}_n\, y)^{i\Uparrow} := \mathsf{X}_{n+1}\, y \qquad (\mathsf{J}_n\, j)^{i\Uparrow} := \begin{cases} \mathsf{J}_{n+1}\, j & (j < i) \\ \mathsf{J}_{n+1}\, (j{+}1) & (j \geq i) \end{cases} \qquad ((M\ N)_n)^{i\Uparrow} := ((M)^{i\Uparrow}\ (N)^{i\Uparrow})_{n+1}$$

which we iterate as $(M)^{i\Uparrow 0} := M$ and $(M)^{i\Uparrow m+1} := ((M)^{i\Uparrow m})^{i\Uparrow}$. As with instantiation, *generalized instantiation*, $(M)[N]^i$, leaves terms M of height 0 unchanged, and updates abstractions:

$$(\mathsf{X}_{n+1}\, y)[M]^i := \mathsf{X}_n\, y \quad (\mathsf{J}_{n+1}\, i)[M]^i := (M)^{i\Uparrow n-i} \qquad ((P\ Q)_{n+1})[M]^i := ((P)[M]^i\ (Q)[M]^i)_n$$

$$(\mathsf{J}_{n+1}\, j)[M]^i := \begin{cases} \mathsf{J}_n\, j & (j < i) \\ \mathsf{J}_n\, (j{-}1) & (j > i) \end{cases}$$

Claim the relation $\bullet > \bullet$ defined without a ξ rule:

$$\frac{\mathcal{W}_{n+1}\, A \quad \mathcal{W}_n\, N}{(A\ N)_n > (A)[N]^n} \qquad \frac{M > M' \quad \mathcal{W}_n\, M \quad \mathcal{W}_n\, N}{(M\ N)_n > (M'\ N)_n} \qquad \frac{N > N' \quad \mathcal{W}_n\, M \quad \mathcal{W}_n\, N}{(M\ N)_n > (M\ N')_n}$$

is equivalent to the relation $\bullet \stackrel{\beta}{\to}_\beta \bullet$ given above (and thus to the usual notion of β reduction).
Proof that $M > N \implies M \stackrel{\beta}{\to} N$ goes by induction on the relation $M > N$. Both congruence rule cases use invertibility of rule ξ for the relation $\bullet \stackrel{\beta}{\to} \bullet$. The converse direction is straightforward. □

Here is Tait–Martin-Löf parallel reduction without a ξ rule.

$$\frac{}{\mathsf{X}_n\, y \gg \mathsf{X}_n\, y} \qquad \frac{n \leq hgt\, M \quad M \gg M' \quad n \leq hgt\, N \quad N \gg N'}{(M\ N)_n \gg (M'\ N')_n} \qquad \frac{j < n}{\mathsf{J}_n\, j \gg \mathsf{J}_n\, j}$$

$$\frac{n < hgt\, A \quad A \gg B \quad n \leq hgt\, M \quad M \gg N}{(A\ M)_n \gg (B)[N]^n}$$

This (nondeterministic) parallel reduction can be made into (deterministic) complete development by replacing the application congruence rule with

$$\frac{n = hgt\, M \quad M \gg M' \quad n \leq hgt\, N \quad N \gg N'}{(M\ N)_n \gg (M'\ N')_n}$$

which removes overlap with the β rule.

Unfortunately this approach doesn't seem to extend to $\beta\eta$ reduction, as rule ξ is not invertible in that case. On this point it is interesting to note that none of the reduction relations in this note can reduce the height of a term, but η reduction can do that.